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ON THE STRENGTHS
OF
BEAMS. COLUMNS & ARCHES

BY

B. BAKER.

ON THE STRENGTHS

OF

BEAMS, COLUMNS, AND ARCHES;

CONSIDERED WITH A VIEW TO

DERIVING METHODS OF ASCERTAINING THE PRACTICAL
STRENGTH OF ANY GIVEN SECTION OF BEAM, COLUMN, OR ARCH;
IN CAST IRON, WROUGHT IRON, OR STEEL.

Compiled partly from Papers contributed to "ENGINEERING."

BY

B. BAKER, ASSOC. INS. C.E.

AUTHOR OF "LONG-SPAN RAILWAY BRIDGES," ETC.

> "Sine me vacivum tempus ne quod dem mihi
> Laboris."
>
> HEAUTON, I. i. 38.

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PRINTED BY C. WHITING, BEAUFORT HOUSE, 30, DUKE STREET,
LINCOLN'S-INN-FIELDS.

PREFACE.

THE papers from which this volume is mainly compiled, may be looked upon rather as expansions of some of the problems most frequently occurring in ordinary Engineering practice, than as attempts at exhaustive theoretical generalisations. The internal evidence of the investigations is sufficient to show that, in most instances, the subjects have been suggested by the varied works of Mr. FOWLER, Past President, Inst. Civil Engineers. In the treatment of the several questions brought under consideration, the Author has endeavoured, in all instances, to assimilate the process of investigation to the ordinary routine of the drawing office: in other words, he has preferred compasses to equations, and scales to logarithms, whenever the selection was optional. In Engineering, lengthy infinitesimal calculations not only involve an unjustifiable waste of time, but have the great contingent disadvantage of checking the growth of sound judgment in the Engineer, by giving a fictitious appearance of accuracy to results which are not susceptible of exact deduction.

The application of mathematics to Engineering problems has attained a highly developed stage; indeed, those conversant with the technical literature of France and Germany, will at once concede that it has far outstripped our present knowledge concerning the strength of materials. We have ample stores of experiments, to enable us as practical men to give safe and economical proportions to our structures; but we have not such a class of experimental data as would enable any mathematician to advance general theories capable of universal application. The practical result of this condition is, that the laborious investigations of the ablest mathematicians commonly resolve themselves into minutely exact formulæ, based upon purely provisional and undemonstrated hypotheses. We are professedly furnished, as it were, with the precise positions of the tangent points in our curves and changes in our gradients, at a time when we have nothing but the roughest possible survey of the route of our line.

The structures considered in the present volume are peculiarly illustrative of the great hiatus in our empirical data. For, although the Author considers it feasible, and has himself in the following pages attempted to advance formulæ which will give practically correct results; yet he is of opinion that without an extensive and special set of experiments arranged for the purpose, it would be perfectly futile to attempt the development of a comprehensive theory to include all cases of beams, columns, or arches.

This short preface is required in justification of certain passages in the following pages, which may possibly appear unphilosophical to such readers as do not admit, with the Author, that there is nothing more fallacious than facts except figures.

B. B.

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THE STRENGTH OF BEAMS.

PART I.

ON THE TRANSVERSE STRENGTH OF STEEL RAILS AND OTHER ANALOGOUS BEAMS.

Section 1.—INTRODUCTORY.

It cannot be disputed that iron rails are rapidly being shelved alongside cast-iron wheels, cork buffers, and other curiosities of railway plant. By a species of natural selection, the better material surely, if but slowly, takes the place of its predecessor. It is no good attempting to stem the current of progress; one must move with the stream, or be swept away by it. The pages of our leading technical journals afford ample evidence that the engineering world generally is fully alive to the fact that steel rails are, in all instances where heavy traffic has to be encountered, destined to supersede iron rails as rapidly as the latter superseded the original type—the cast-iron tram-plate.

Such being the case, now is the proper time to endeavour to deduce results from the valuable mass of experiments which have been carried out at different times and places on the strength of steel rails. Without such data, although we knew the quality of steel employed, we should still be ignorant of the strength of a rail of any particular section till we had rolled one and tested it; but with the evidence already accumulated we think there should be little difficulty in predicting very closely the behaviour of any piece of rail under any given test. The object of the present book is to show how this may be done; it is a problem which any engineer may any day have to work out for himself, since the sections of rails even of similar weights per yard are very varied, and of rails of different weights the numbers are almost infinite. The strengths of these rails will, of course, be comprised within very wide limits only; and in order to arrive at the comparative qualities of the steel used in the manufacture of the several rails, the tests applied should be carefully determined upon certain fixed principles which we shall now endeavour to develop.

The first thing to be considered is the nature of the test to be applied, and we can state without any reserve that nothing is, in our opinion, so good as the simple application of a gradually increasing bending stress to the centre of a piece of rail resting upon bearings about 5 ft. apart, the deflection and set being noted after each successive increment of stress. The rough and ready test of a falling weight is inexpensive, and it answered well enough in its day; but for comparative results it is, as we shall shortly show, a very fallacious guide. The majority of works are unprovided with monkeys of sufficient power to smash a good steel rail, and it is very questionable whether it is desirable that they should erect them, since, if they do not, the rails will, in all probability, be sent to some testing works, where they will undergo a more rational mode of treatment. Some engineers support a falling weight as a test of toughness, and this argument is plausible enough at first; but a more detailed examination will show that the same results may be obtained in a *measurable* form by the system of transverse strains with observed deflections.

Thus, with a falling weight, the *work done* in fracturing a rail—assuming the weight of the specimen to be *nil*, and that of the anvil-block infinite—is simply the product of the weight of the monkey by the height from which it falls. If the conditions of immovable anvil and weightless specimen could be obtained in practice, falling weights and transverse strains would be convertible terms. The *work done* in each instance would be the product of the ultimate deflection by half the ultimate transverse strength, the fraction $\frac{1}{2}$ being a sufficiently near approximation to the mean bending force exerted in terms of the maximum, since it is obvious that both the force and the deflection would commence at zero. From the preceding equation we could, if the conditions assumed were attainable, easily determine either the weight of monkey or the height of fall, the other conditions being given. We will now show by a practical example the variance between theory and practice in such instances:

A piece of rail 4 ft. long was placed on bearings 3 ft. apart, and a weight of 1 ton was allowed to fall on the centre of it from a height of 30 ft., the result of which was a bending of the rail to the extent of .6 ft. At another place, a different section of rail was fixed at the same bearings, and tested in a similar manner, when it was found that a fall of 23 ft. sufficed to produce the deflection attained in the other specimen with a fall of 30 ft. The *work done* in producing these deflections was, therefore, 30 foot-tons and 23 foot-tons respectively, amounts in each instance far in excess of those theoretically required. Thus the deflection of the first rail under the blow was that due to a weight of 40 tons applied at the centre of the 3 ft. bearings, and the *work done* by a gradually increasing bending strain of that intensity in producing the given deflection would be:

$$\frac{40 \text{ tons} \times .6 \text{ ft. deflection}}{2}$$

$=12$ foot-tons. The deflection of the second rail was that due to a stress of 44 tons at the centre, and the work done would, therefore, be

$$\frac{44 \text{ tons} \times .6 \text{ ft. deflection}}{2}$$

$=13.2$ foot-tons. In the first experiment it appears that the actual test applied to the rails was but 40 per cent. of that which it might, upon superficial consideration alone, have been supposed to have sustained; whilst in the second experiment it was about 60 per cent. In these two simple experiments, then, a discrepancy of 50 per cent. was evidenced, and in all probability that was due to the fact of the rails being tested at different places and under different monkeys. The enormous excess of power required to bend these rails by a falling weight beyond the amount necessary to produce the same effect by a steady bending stress cannot be ascribed to the influence of the *inertia* of the specimens. The formula advanced by Tredgold, and confirmed by Hodgkinson, for the deflection of beams under horizontal impacts is as follows:

$$\delta = \sqrt{\frac{2\,h\,W^2}{P\,(W+w)}}$$

h being the fall in inches corresponding to the velocity of impact; W the weight of falling body; w that of the specimen reduced to the equivalent amount applied at the centre of the span, and P the stress required to bend the beam 1 in.

It appears then that the deflection is reduced by the influence of the *inertia* of the specimen to the extent represented by the fraction $\frac{w}{W+w}$. In the instance of the rails we have referred to, the weight of the specimen was far too small in relation to that of the falling weight to afford any explanation of the great anomaly exhibited.

With any amount of such evidence available, we think it will be obvious to all how much more satisfactory and reliable results must be when the uncertain elements, inseparable from every test conducted by means of a falling weight, are excluded, and the *work done* is accurately estimated by direct observations of deflections and transverse strains. The results in the latter case are, after all, identical with those which would be obtained in the former, if all complicating conditions could be eliminated. The amount of "fatigue" any rail can endure will always be directly proportional to the product of the ultimate deflection and the ultimate transverse strain.

This principle is at the present time well understood and acted upon in testing bars by direct tension. No engineer of experience would attach the slightest weight to experiments on the ultimate resistance to tension *alone* as determining the quality of metal, since it is well known that the most trashy iron often gives very high results. To arrive at the real value of the material, it is absolutely necessary to know the *work done* in producing fracture, and that, as in the instance of the steel rail, will always be proportional to the product of the ultimate tensile resistance by the ultimate extension; and the toughest and best material will always give the highest result, if compared with others on that system.

It is, of course, necessary in judging of the quality of steel, or other materials, by direct experiment to keep in mind the influence the *form* of the specimen often exercises. By injudicious shaping, the best material may be reduced to the level of the worst. As a case in point, we may instance the injury done to a bar of iron by screwing it. Some of our readers may remember the general result of the experiments on Palliser's bolts, which were designed to obviate this injurious effect of screwing by reducing the sectional area of the shank to the same amount as that obtained in the screwed portion. The amount of "fatigue" these bolts underwent before fracture was fully three times as great as that sustained by ordinary bolts of the same size and of the same quality of metal; and it follows, as a corollary, that if the results had been similar, the quality of metal, instead of being also similar, as might at first be supposed, must in reality have been far above the average in the old, and far below it in the new, style of bolt. Again, it is obvious we may "nick" a steel rail till we reduce its power of resisting a blow to the level of that of a cast-iron rail; we shall have to consider this more fully hereafter in deducing the effect of punching on the practical strength of steel flanged rails.

If all such disturbing influences are duly eliminated, there will be no necessity for any other test of the quality of the steel employed in the manufacture of rails than a carefully conducted series of experiments on the rail under a gradually increasing bending stress. To enable us to attain that desirable result, it will be well to consider generally the nature of transverse strain as far as it affects the half-girder and half-solid section to which rails are commonly rolled.

Section 2.

ON TRANSVERSE STRAIN.

For a considerable period the idea was entertained by practical men, and in a few instances it is even yet maintained, that the skin of a bar is by far the stronger part of it. This prevalent idea of the workshop was apparently borne out by theoretical deductions from careful experiments, since mathematicians stated that an anomaly was presented which the assumption of excessive strength in the skin would clear up. The matter was thus left in abeyance for a long time, although, as it is hardly necessary now to remark, the apparent coincidence arose from entirely different causes. The idea of the strength of the skin probably arose from the disastrous results which followed the cutting of it, not on account of its greater strength per unit of area, for experiments on turned bars show this to have no foundation in fact, but on account of the concentration of the deflection, or extension, at the point of reduced sectional area. Again, the waves of vibration are interrupted at the same spot, the inevitable result of which is, of course, that the limit of elasticity is frequently passed, and the metal consequently deteriorated piecemeal till it can no longer offer any effective resistance. It remained for Mr. W. H. Barlow more fully to elucidate the question; and in the Transactions of the Philosophical Society for 1855 and 1857, will be found the development of his theory of the existence of an element of strength in beams, which he has denominated "the resistance of flexure."

In the investigation of problems relating to the transverse strength of beams, the commonly accepted law, *ut tensio sic vis*, is a sufficient approximation to the truth in instances where the "web" portion is small in comparison with the gross section; but when, as in a solid square or rectangular bar, it is of equal section per unit of depth with the "flange" portions, or still more, when, as in a round bar, or in a square bar strained in the direction of a diagonal, the web portion is even more excessive, that simple law gives perfectly ridiculous results. In fact, where the material is cast iron, the error in a square bar, strained in a direction parallel with one of its sides, is 143 per cent.; in a round bar it is 174 per cent.; and in a square bar strained in the direction of one of its diagonals, the error is no less than 190 per cent.!

The preceding results deduced from experiments on cast iron, although apparently not bearing directly on the question we are considering, afford strong presumptive evidence that a similar divergence from the ordinarily accepted law will obtain in steel beams, and the necessity of investigating this point before we can hope to arrive at any fair estimate of the resistance to be expected from a given section of steel rail will be at once evident.

When investigating the theory of the resistance of materials to a transverse strain, Mr. Barlow was struck by the consideration that, in applying the law, *ut tensio sic vis*, to contiguous fibres, the effect of *lateral adhesion* was omitted, since each fibre was assumed to be capable of taking up the same degree of extension and compression from the same force as if it acted separately and independently of the adjoining fibres. Now, it is a well-known practical fact that a powerful lateral action does in reality exist, tending to modify the effect of unequal strains. This fact was proved by Mr. Barlow beyond suspicion by direct measurements of the extensions and compressions of the outer fibres of beams under certain transverse strains. Thus, in a beam 7 ft. 4 in. long, 6 in. deep, and 2 in. thick, the extension of the outer fibres under a calculated strain of 14,666 lb. per square inch was $\frac{1}{1207}$ of the length, whilst under a direct tensile strain of similar amount the measured extension was $\frac{1}{645}$ of the length, or nearly double that resulting from the same strain per square inch on the metal when in the form of a beam tested by transverse strain. There can be no question, therefore, that lateral action does modify the effect of unequal strains on contiguous fibres, and that the deflection of a beam, as well as its ultimate strength, is governed by this condition to so great an extent as to make the neglect of it result in an error of any amount up to 190 per cent. This new resistance will be dependent upon the amount of *deflection* in connexion with the *depth* of metal in the beam, and it will be uniform per unit of area over the whole cross section, since it is due to the elastic reaction of the contiguous fibres, which will be a constant amount throughout the whole depth of the beam. This, of course, premises that the extension and compression of the several fibres will be directly proportional to their distance from the neutral axis—a fact well ascertained to be the case by direct experiment.

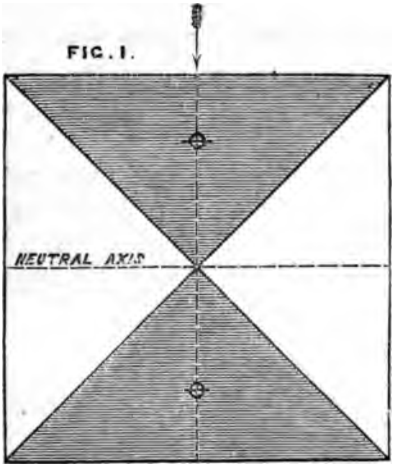
It follows, then, that whatever the form of cross section of a beam, the value of the ultimate resistance of the fibres, as deduced by mathematical analysis, will be higher than the resistance of the material to direct tension. Adopting Mr. Barlow's notation, f being the ultimate resistance to direct tension, F the *apparent* resistance to the same force excited by transverse strain, and φ the resistance due to flexure, then $F = f + \varphi$. The variable quantity in this equation is φ, which will be governed by the form of cross section. For our present purpose, we need not complicate the questions under consideration by any general and comprehensive formulæ; nor, indeed, is it all proved that in any instance the difference between the apparent and actual strain on the fibres of a beam is due to the lateral action alone, or that any theory yet advanced can be accepted unreservedly. For practical purposes it will only be necessary to ascertain the values of the several symbols in solid bars, which may be easily done by availing ourselves of some of the numerous experiments on the transverse strength of that class of specimens, when the application of the results to rail sections will follow easily. For it will be shown that, even in extremely complicated forms of cross section, the value of φ may be at once obtained from its ascertained value in the solid bar by simple inspection; and that computations of transverse strength, based upon the system advanced, give results, in each instance, practically identical with those derived from direct experiment.

Before we can do this, however, it will be necessary to consider some method of ascertaining the *moment of resistance* of the given cross section; that is, the product of the effective area of metal on one side of the neutral axis by the effective leverage at which it acts, or, in common engineering parlance, the area of flange × depth of the girder. Having this moment of resistance, M, and also the moment of the weight, m, that is, the bending force multiplied by the effective leverage at which it acts, we may at once obtain the value of F, since obviously $F = \frac{m}{M}.$ The value of m is readily arrived at, since it is merely the moment of the weight round the support. Thus, with a weight W, applied at the centre of a given span, l, the moment of the weight round one support will be $\frac{W}{2}$ for the effective weight, and $\frac{l}{2}$ for the effective leverage, therefore $m = \frac{Wl}{4}$; and under all conditions of loading it is essential to the condition of stability that the moments of the weights round each support should be equal. With an equally distributed load, the centre of gravity of the portion transferred to each support being distant $\frac{l}{4}$ from that point, the effective leverage will be equal to that amount, and the moment, $m = \frac{Wl}{8}$. Again, if the weight W be applied at a point dividing the span, l, into the given segments, a and b, the moment will be the weight multiplied by the product of the two segments and divided by their sum, that is, $m = \frac{Wab}{l}$. Under any given distribution of load the value of m can be easily obtained by the method of commencing at each support simultaneously, and multiplying the successive increments of weight by the distance of their respective centres of gravity from the supports; carrying on the process till the whole of the weights have been dealt with, and the moments on each side balance.

To determine the moment of resistance of any given cross section is generally a rather more tedious process than the foregoing, but it is equally simple, although, from the fact of the calculus almost always figuring in the portions of engineering works treating on the subject, most untheoretical men would naturally be led to infer the contrary. We shall show how the required result may be obtained with very few figures, or indeed, if necessary, without any figures at all.

The determination of the moment of resistance of a given cross section is based upon the theory that every fibre in a beam is strained in direct proportion to its distance from the neutral axis, and that the effective leverage at which each fibre acts is also proportional to the same amount. Representing the effect of this condition, geometrically, for a square cross section we are informed of the following facts. The neutral axis being at the middle of the depth, at that point the horizontal strain will be *nil*, increasing to the maximum amount at the extreme depth. If, therefore, we draw the diagonals of the square (Fig. 1), we obtain two triangles

FIG. 1.

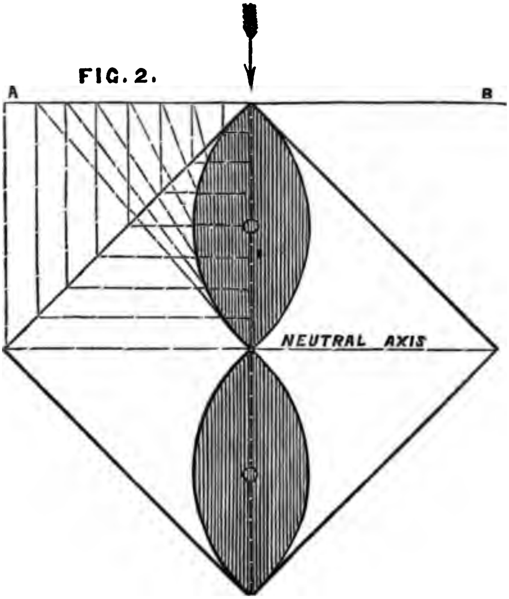


—shaded on the diagram—the width of which, at any given distance from the neutral axis, will be proportional to the horizontal strain on the fibres at the same point, and the *area* of which will, consequently, be proportional to the sum of the horizontal action of all the fibres on each side of the neutral axis. In short, the area of the triangle represents the equivalent area of metal, assuming the stress at all points to be equal in amount to that on the extreme fibres. Having the effective area of *flange*, it is necessary to ascertain the effective depth. Now, as the triangle represents the distribution of forces of equal intensity per unit of area, the centre of pressure will obviously be at the same point as the centre of gravity of a triangle; that is, at two-thirds the height of the triangle from the apex. The distance from centre of pressure of one triangle to the same point in the opposite triangle will, consequently, be the effective depth, and be equal to $\frac{2}{3}d$, when d is the side of the square. Again, the area of each triangle being equal to $\frac{d^2}{4}$, the moment of resistance of a square cross section will be

$$\frac{d^2}{4} \times \tfrac{2}{3}d\text{; that is, } M = \frac{d^3}{6}.$$

The process will be more clearly illustrated if we take the more complex problem of a square bar strained in the direction of a diagonal, which may be solved geometrically as follows (Fig. 2): Draw the diagonals of the

FIG. 2.



square, and divide the one in the plane of which the bending force acts into any number of equal parts. Through the points thus set off draw lines at right angles, intersecting the sides of the square, and from the points of intersection set up verticals cutting the line A — B. Remembering that the strain on each fibre is proportional to the distance from the neutral axis, it will be at once seen that if we draw lines from the points thus obtained on the line A — B to the neutral axis, the intersection of these lines with the horizontal ones, representing the several layers of fibres, will give the equivalent lengths of fibres, if the same intensity of force were acting throughout; and, consequently, by joining the several points, we obtain a figure, shaded on diagram, representing the sum of all the forces, and the area of which will be the effective area of "flange."

The area of this figure might be found by direct measurement from the diagram, and its centre of gravity in the same way; but in this instance it will be seen by inspection that the shaded space is a parabola, since the ordinates vary as the rectangles of segments. The area of a parabola is $\frac{2}{3}$ of the circumscribed rectangle, consequently the effective area of the shaded portion will be

$$\frac{2}{3} \times \frac{d}{2\sqrt{2}} \times \frac{d}{\sqrt{2}};$$

and the centre of pressure being at its centre, the effective depth will be $\frac{d}{\sqrt{2}}$; and the moment of the cross section

$$= \frac{2}{3} \times \frac{d}{2\sqrt{2}} \times \frac{d}{\sqrt{2}} \times \frac{d}{\sqrt{2}},$$

therefore

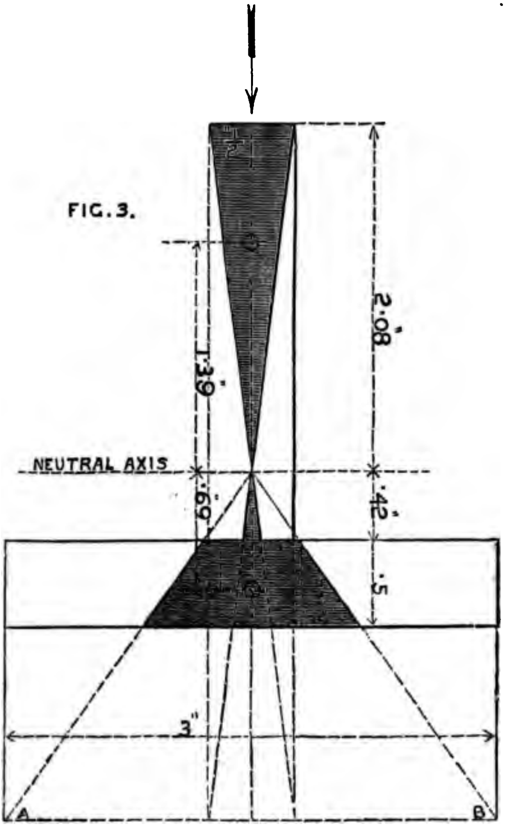
$$M = \frac{d^3}{6\sqrt{2}}.$$

It appears, then, that the mathematical, though not as we shall hereafter show the practical, strength of a square beam strained in the direction of one of the diagonals of its cross section, is less than that of the same section strained in a direction parallel to one of its sides in the proportion of $\sqrt{2}$ to unity, that is, as 1.41 : 1.

It will be well now to give an example of the application of the preceding method to beams of non-symmetrical cross sections, and we cannot choose a better section for the purpose that an ordinary piece of T-iron 3 in. × 3 in. × $\frac{1}{2}$ in.

In this instance the position of the neutral axis, which in a beam of symmetrical cross section is at the middle of the depth, has to be determined. With a little practice, the position which will correspond with the centre of gravity of the cross section, may be fixed with a very considerable degree of accuracy by inspection merely; having thus assumed some position for it, the construction of the diagram will be as follows : It is premised that the maximum strain per unit of area is the problem to be solved; the strain, therefore, on the extreme fibres of the vertical web will be the measure of the strength of cross section. Set off, then (Fig. 3), from the neutral axis a point, at the same distance from it, as the extreme fibre of the vertical web, and through that point draw the line A—B, squaring down to that line the edge of the bottom table and the thickness of web of the T. From the points thus obtained draw lines radiating towards the neutral axis, then the space enclosed by these lines (shaded on the diagram) will represent the equivalent sectional area,

FIG. 3.



if strained to the same intensity as the extreme fibre on the opposite side of the neutral axis. On that side, of course, the figure will be a simple triangle. Now, since these shaded areas represent the distribution of forces of equal intensity per unit of area, and since action and reaction are equal and opposite with respect to the neutral axis, it follows, if the position of that point has been correctly assumed, that the areas of the shaded portions above and below it must necessarily be equal. If, as is most probable, they should not balance, it would be necessary to assume a fresh position for the neutral axis, and generally the second trial will give the position sufficiently near the truth for all practical purposes. The degree of accuracy with which the theoretical position of the neutral axis in complicated forms of cross section may be ascertained by this method is very much greater than would at first be imagined; even a difference of but $\frac{1}{100}$th of an inch in its position is clearly indicated in a magnified form by the process of computation.

Carrying out the method thus indicated on the specimen of T-iron selected as an example, we find the position of the neutral axis to be 2.08 in. from the end of web, the effective area of metal being .52 square inch above and below that point.

It is necessary now to determine the effective leverage at which this metal acts, or, in other words, the effective depth of the T-iron; and, from what we have already observed with reference to previous sections, it will be seen that this depth will be represented by the distance between the centres of gravity of the shaded portions on either side of the neutral axis. Now, the centre of pressure of the upper portion measured from the neutral axis will obviously be $\frac{2}{3}$rds of the distance from that point to the extreme fibre, that is, $\frac{2}{3} \times 2.08 = 1.39$ in.; and the centre of pressure of the opposing portion will be equal to the moment of it round the neutral axis divided by the effective area, which we have already found to be .52 square inch. The required calculation to determine the centre of pressure from the diagram is *as follows*:

$$\begin{array}{llll} & \text{Area} & \text{Leverage.} & \text{Moment.} \\ \text{Portion of web} & .02 \times & \left(\frac{2}{3} \times .42\right) & = .0056 \\ \text{Flange} \ldots\ldots \left\{ \begin{array}{l} \\ \\ \end{array} \right. & \begin{array}{l} .19 \times \\ .31 \times \end{array} & \begin{array}{l} \left(\frac{2\times1}{3\times2} + .42\right) \\ \left(\frac{1\times1}{2\times2} + .42\right) \end{array} & \left.\begin{array}{l} = .1425 \\ = .2077 \end{array}\right\} = .356 \end{array}$$

The moment, .356, divided by the area, .52, will, as we have already remarked, give the centre of pressure of the figure below the neutral axis, which will therefore, be

$$\frac{.356}{.52} = .69 \text{ in.}$$

We have already found the centre of pressure of the upper portion to be 1.39 in. from the same point; the total distance between the centres of pressure will consequently be $1.39 + .69 = 2.08$ in.; and the moment of resistance (M) of the 3 in. × 2 in. × $\frac{1}{2}$ in. T-iron, being the product of that distance into the effective area, we have $M = 2.08 \times .52 = 1.08$.

The value of M being thus deduced, we can at once obtain the maximum strain on any fibre under any given transverse strain; and should it be desirable to know the maximum strain per unit of area, on the lower flange of the T-iron, as might be the case if the material were cast iron, it could be at once obtained from the preceding construction, since it would be less than the maximum strain on the extreme fibres in direct proportion to its comparative distance from the neutral axis, that is, as $(3 - 2.08) = .92$ in. : 2.08 in.

Having thus illustrated a method by which we are enabled to ascertain with facility and accuracy the mathematical value of any given cross section, symmetrical or otherwise, we are in a position to analyse the results of direct experiments on transverse strength, and to apply the data so obtained to the elucidation of the question we are considering, namely, the determination of the transverse strengths of steel rails of any given section. There are, of course, many other methods by which we could ascertain the moment of resistance of a given cross section, but the preceding geometrical process will, we think, be found the most generally advantageous for ordinary office use. The several steps in the process are kept before the eye, errors of any moment are thus rendered practically impossible. The point at which our investigation of the transverse strength of beams was stopped, was, it will be remembered, when it became necessary to ascertain the values of the symbols in the equation $F = f + \varphi$. We can readily do this, now we are enabled to determine the moment of resistance of any bar experimented upon, since, as we have already observed, the value of F is simply the moment of the weight divided by the moment of the cross section; and knowing the resistance to tension, f, by direct experiment, it is obvious we have also the value of φ.

In deducing the required data from experiments on transverse strength, our subject will be rendered clearer, if we include experiments on cast and wrought-iron as well as on steel. Although apparently not bearing directly upon the question, the comparison will be valuable, as indicating the direction in which modifications may be looked for according as the steel employed more nearly simulates cast or wrought iron in its attributes of hardness or toughness. We shall therefore first avail ourselves of the valuable series of experiments by Hodgkinson, Barlow, and others, on cast iron.

Section 3.

ON THE STRENGTH OF CAST IRON.

Although, in estimating the necessary section of iron girders and other structures, the kind of iron proposed to be employed seldom enters into the calculation, the unit strain being fixed irrespective of that condition, yet, as a matter of fact, it is well known that the variation in the strengths of different specimens of iron amounts sometimes to more than 100 per cent. Thus, in ten different descriptions of cast iron experimented upon by Mr. Hodgkinson, the tensile strength varied from 12,694 lb. to 23,468 lb. per square inch, the average being 15,711 lb. per square inch; and the transverse strength of bars 1 in. square, and 54 in. between the supports, ranged from 357 lb. in the weakest sample to 581 lb. in the strongest, the mean being 450 lb. Within certain limits remelting increases the strength of iron, in some instances as much as 75 per cent. transversely, and no less than 133 per cent. under a direct pull. The great tensile resistance of 38,000 lb. per square inch, attained in American cannon iron is partly due to this cause.

Deducing the value of φ from the mean results of Mr. Hodgkinson's experiments on the class of iron most commonly used, we have $f = 16{,}502$ lb., and $F = \frac{m}{M}$, when $M = \frac{d^3}{6} = \frac{1}{6}$; and since the weight was applied in the centre of the bearing

$$m = \frac{54 \text{ in.} \times 464 \text{ lb.}}{4} = 6264 \text{ lb.}$$

Hence $F = 6 \times 6264 = 37{,}584$ lb.; and $\varphi = F - f = 36{,}584 - 16{,}502 = 21{,}082$ lb. The ratio of the function φ to f in these experiments, then appears to be as 1 : 78.

The experiments carried out by Mr. Barlow on cast-iron bars of various sections corroborate the preceding result, as will be apparent from the following analysis. The sections tested were rectangular, circular, and squares placed diagonally. For the former section the value of the moment of the cross section will, of course, be $M = \frac{d^2 b}{6}$; and for the latter we have found it to be $M = \frac{d^3}{8.4}$. By the same method we might have shown that for a circular cross section $M = \frac{d^3}{10.2}$; and, since the weights were in all instances applied at the centre of the span of 60 in., the moment of the weight in each experiment would be $m = 15\,w$. Tabulating the results of these experiments, we obtain the following values of F; and since the mean tensile strength (f) of the iron employed was found to be 18,750 lb. per square inch for the smaller and 16,800 lb. for the larger sections, we are also enabled to deduce the values of φ.

Form of Cross Section.	Breaking Weight.	Value of F.	Value of φ.
	lb.	lb.	
Rectangular, 2 in. deep × 1 in........	1888	41,709	22,959
Square, 1 in. × 1 in.....................	527	45,630	26,880
Round, 1 in. area........................	474	51,396	32,646
Square, 1 in. broken diagonally......	449	53,966	35,216
Square, 2 in. × 2 in.....................	3478	39,094	22,294
Round, 2½ in. diameter.................	4143	39,560	22,760
Round, 2¼ in. diameter.................	3132	44,957	28,157
Square, 2 in. broken diagonally......	2988	47,746	30,946

Averaging the preceding results, and correcting them *by comparison* with those previously deduced, and with the results of other experiments which it is not necessary to detail here, we find the several values of φ for the different forms of cross section experimented upon to be as follows:

Rectangular cross-section . . . . . . $\phi=1\frac{1}{4}f$.
Round ,, ... $\phi=1\frac{1}{2}f$.
Square (in direction of diagonal). . . $\phi=1\frac{1}{4}f$.

It is shown, then, by direct experiment, as well as by theoretical investigations of the resistance of flexure, that the nearer the centres of gravity of the portions of cross section above and below the neutral axis approach that point, the higher will be the value of φ; and of course, as the converse will hold good, in a girder section, where the centre of gravity approaches nearly the extreme points of depth, the value of φ will be a minimum, and may in most practical cases be neglected, as customary, without material error.

As practical engineers almost universally proportion the several portions of iron structures to a certain predetermined strain per square inch on the metal, a taint of empiricism appears to attach to ordinary formulæ for solid beams in which the unit strain does not figure. Since we have now before us the several values of φ we are enabled to calculate a solid beam by precisely the same method which we should adopt were it an ordinary girder, as will be seen by the following example:

Required the section of a girder 12 in. deep, and also of a solid round shaft, to carry a load of 10 tons at the centre of a span of 16 ft., the strain being 2 tons per square inch.

The weight being applied at the centre of the span, its moment will be

$$m=\frac{16'\times 12''\times 10}{4}$$

$=480$; and for the girder section the area of flange would, consequently, be:

$$a=\frac{480}{12''\text{ deep}\times 2\text{ tons per square inch}}=20\text{ square inches.}$$

Now for the round shaft we have the given strain $f=2$ tons per square inch, and we have found the value of ϕ for a round section to be $1\frac{1}{2}f$; hence $\phi=3$ tons, and $F=f+\varphi=5$ tons per square inch. That is, with an actual strain of 2 tons per square inch on the extreme fibres of the shaft, the *apparent* amount, deduced mathematically, would be 5 tons per square inch; we have, therefore, to substitute that amount for the original 2 tons per square inch in calculating the necessary diameter of the shaft. Since $m=480$, and $F=5$, in the equation $m=M\,F$, the moment of the cross section required will be

$$M=\frac{m}{F}=\frac{480}{5}=96;$$

and since in a circular cross section we have found

$$M=\frac{d^3}{10.2},\text{ we have }d^3=96\times 10.2,\text{ hence }d=10\text{ in.}$$

diameter nearly.

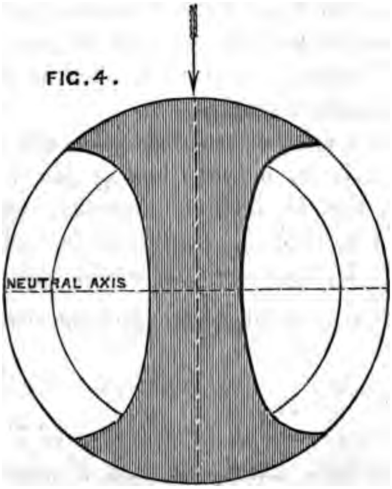
The importance of duly considering the essential difference between the *actual* and *apparent* strain on the metal in beams of more or less solid forms of cross section, when proportioning the several parts of any cast-iron structure, cannot be over estimated. Thus, if we had to deal with a hollow shaft, theoretically, its resistance would be proportional to the moment of resistance of a solid shaft of the same diameter as the exterior of the tube multiplied by the maximum unit strain on the metal, less the moment of resistance of a solid shaft equal to the internal diameter of the hollow shaft, also multiplied by the maximum unit strain on the material. Now, we have shown the moments of resistance of circular cross sections to be proportional to the cubes of their diameters, and the unit strains at different points of the section to be proportional to the distance from neutral axis, hence the maximum unit strain on the portion of metal removed in forming a hollow shaft would, if *in situ*, be less than that on the extreme fibres in the same ratio as the respective diameters. It follows, then, that D being the exterior diameter of a hollow shaft, and d the interior diameter of the same, its theoretical strength in terms of a solid shaft of the diameter D would be $\frac{D^4-d^4}{D^4}$.

Upon this conclusion most formulæ for the strength of hollow shafts, as given in our ordinary text books, are based. From what we have already advanced, the fallacy of results so deduced may be predicated, and we will now illustrate the practical effect of the error involved by a simple example.

Required the ultimate transverse strength of a hollow shaft, the thickness of metal being $\frac{1}{8}$th the external diameter D, that is, internal diameter : external diameter :: 6 : 8. Then, in terms of the solid shaft of the diameter D, the strength would theoretically be $\frac{8^4-6^4}{8^4}=\frac{2800}{4096}$; or about $\frac{7}{10}$ths. But the strength of the solid shaft will be $F\frac{d^3}{10.2}$, so in order to solve our problem, it is only necessary to know the value of F; and it is owing to the false assumption that F would have the same value for hollow as for solid shafts that ordinary formulæ are at fault.

Now, if we distribute the metal forming the hollow shaft, in the same relative position with respect to the neutral axis, and therefore offering the same resistance to transverse strain, but in the form of a solid cross section, we obtain the beam shown shaded in Fig. 4. It will be evident at a glance that this cross section approximates more nearly to a girder than to a rectangular section, and still less to the more advantageous circular section. In fact, experiments prove that the value of φ in the equation $F = f + \varphi$, which, for a rectangular cross section we have found to be $1\frac{1}{4}f$, in this cross section would be less in about the ratio of the minimum thickness of the beam — at the neutral axis — to the maximum width near the extreme points of depth. In the present example that ratio is $\frac{3}{8}$, and, consequently, the value of φ would be $\frac{3}{8} \times 1\frac{1}{4} = \frac{15}{32}f$.

FIG. 4.



It follows, then, that with a working strain equivalent to 2 tons per square inch, as previously assumed in the calculation of the strength of the solid shaft carrying a load of 10 tons in the centre of 16 ft. bearings, the mathematical value of the strain at the extreme fibres would be $1\frac{15}{32} \times 2$, or about 3 tons per square inch; whilst in the instance of the solid shaft it may be remembered it had the value of 5 tons per square inch, thus showing an error of 66 per cent. as resulting from the application of the ordinary formulæ, in which the variable value of φ is not considered. In other words, the strength of the hollow shaft with metal $\frac{1}{8}$th the major diameter in thickness, instead of being $\frac{7}{10}$ths that of the solid shaft as deduced on the ordinary hypothesis, would in reality be only $\frac{7}{10} \times \frac{3}{5} = \frac{4}{10}$, about, of that amount.

Analogous results would be arrived at were we to compute the transverse strength of a rectangular box, or any other form of beam holding an intermediate position between the solid bar and flanged girder types. Indeed, previous to Mr. Hodgkinson's direct experiments on the strength of flanged girders it was commonly assumed that the strength of such girders would be that of the solid rectangular beam out of which the girder may be imagined to have been carved, less the proportional resistance of the portions removed. By this method the strength of the girder would, of course, be exaggerated some cent. per cent. This error was, as far as ordinary girders are concerned, eliminated by Mr. Hodgkinson's experiments; but where other forms of cross section are involved the original hypothesis is commonly maintained, although the operation of the same laws in all instances is indisputable.

Having thus carefully, though briefly, analysed an adequate number of experiments on the transverse strength of cast iron, the investigation of the behaviour of wrought iron and steel under similar strains will be a comparatively simple process, since it will merely be necessary to deduce the values of f and φ from direct experiments, to enable us to ascertain by the process already exhibited the probable transverse strength of any given cross-section of beam, constructed of either of those materials.

Section 4.

ON THE STRENGTH OF WROUGHT IRON.

Experiments on the direct tensile resistance of wrought iron show the same want of uniformity in the quality of the metal which we have already found to obtain with cast iron. Thus, in 150 pieces of *plate* tested at Messrs. Napier's works at Glasgow, the strength ranged between the limits of 14½ tons and 28 tons per square inch, the variation being about 93 per cent. Again, in 195 *bars* experimented upon at the same place, the ultimate resistance to tension varied from 19¾ tons to 30½ tons per square inch. The most complete series of experiments are undoubtedly those made by Mr. Kirkaldy, who tested upwards of 600 specimens of iron from all districts, with the following *mean* results:

	Ultimate Resistance to Tension.	Reduction of Sectional Area at point of Fracture.
	Tons per sq. in.	Per cent.
Bars....................	24 to 27½	22.6 to 60.9
Angle-irons............	18½ „ 27½	9.2 „ 41.4
Plates lengthways ..	20½ „ 24½	6.9 „ 20.9
Plates (across grain).	18½ „ 22½	8.7 „ 12.8

The experiments of Fairbairn and Clarke on Staffordshire and Yorkshire bridge iron gave results varying from 18 to 22 tons per square inch.

An important experiment on the effect of repeated applications of a breaking strain was carried out by Mr. Fairbairn. It was found, contrary to anticipation, that instead of the metal being injured by four or five applications of a strain producing as many successive ruptures, it was, in reality, improved in its powers of resistance, since, notwithstanding the reduced sectional area of the bar, the final fracture required greater force than either of its predecessors. This fact was corroborated in a very interesting manner by some experiments of Edwin Clarke on the transverse strength of bars previously subjected to a bending force, to which we shall hereafter have occasion to allude.

In the first section of these papers we insisted upon the importance, and, indeed, the absolute necessity, of noting the reduction of area, or, what amounts to the same thing, the elongation of a specimen under the ultimate tensile strain. If this be done, as pointed out by Mr. Kirkaldy, the apparent anomaly of inferior iron bearing fully one-third more steady load than much better material at once vanishes, since the former material maintains its full sectional area nearly up to the time of fracture, whilst the latter has to do the work with a sectional area some one-fourth only of the original amount. Of course the former material would snap with the slightest concussion, whilst the latter would bear almost any amount of ill usage.

In our investigation of the strength of cast-iron beams, the resistance to tensile strains alone were considered, the compressive and shearing strains being entirely ignored. This may safely be done in most practical cases, even when the cross section of the beam is of the true girder type, since it has been amply proved by experiment that the ultimate compression sustained by cast iron is from 4½ to 6½ times the amount of tension the same sectional area would support. Practical considerations generally make it desirable to give a larger area to the parts in compression than those ratios would dictate, and the consequence is, that in cast-iron beams, as previously stated, it is seldom necessary to consider the specific strains on those parts.

Now for a wrought-iron beam of symmetrical cross section—in this country at least—the resistance of the material to compression is almost universally taken as the element governing the strength. The ultimate resistance to compression is considered to be only about ⅓rd that of cast iron, or 16 tons per square inch. This assumption may be necessary, and at all events it is a safe one to make, when dealing with wrought-iron girders of the ordinary proportions; but in the present investigation, where cross sections more or less solid have to be considered, it would lead to false conclusions. It is true the elastic resistance of the metal is overcome by about the same strain per square inch, whether it is acting in compression or tension, and on that account many continental engineers make the top and bottom flanges of wrought-iron girders of equal area, instead of giving a preponderance to the top flange of some 20 per cent. as customary here; yet the ultimate resistance of a cube of wrought iron to compression is almost unlimited. Thus a piece of hammered Swedish iron, the elastic strength of which was about 11 tons per square inch both in compression and tension, bore upwards of 66 tons per square inch compression with a resulting depression of about 45 per cent. The ultimate tensile strength of the same iron was considerably less than $\frac{1}{3}$rd of that strain, whilst the *extension* of the *centre* portion of the bar was fully as great as the *depression* under the larger strain.

It is palpable, then, that a bar of such material under transverse strain would be "set up" on the compressed side, and "drawn out" on the opposite one, the sectional area of metal resisting compression increasing in nearly the same ratio as that in tension decreases. Coupling this fact with the experiment cited, it requires no demonstration to show that for wrought-iron solid beams of symmetrical section, as well as for cast-iron ones, and as we shall hereafter find for steel ones also, we may safely confine our attention to tensile strains only. We may, therefore, at once proceed to deduce the values of f and φ from the results of actual experiment.

Taking, first, some recent experiments by Kirkaldy on hammered Swedish iron bars, 2 in. square, the mean tensile strength was found to be 42,133 lb. (18$\frac{3}{4}$ tons) per square inch; and the breaking weight at the centre of 25 in. bearing averaged 14,000 lb.

Now, the moment of the weight

$$=m=\frac{25\text{ in.}\times 14{,}000\text{ lb.}}{4}=87{,}500\,;$$

and the moment of resistance of the cross section

$$\text{M}=\frac{2^3}{6}=1\tfrac{1}{3}.$$

Hence, the apparent strain on the extreme fibres

$$\text{F}=\frac{m}{\text{M}}=\frac{87{,}500}{1\frac{1}{3}}=65{,}625\text{ lb. } (29\tfrac{1}{4}\text{ tons) per square in.}$$

Resolving F in toits components, f and φ, we have, since $f=42{,}133$ lb., $\varphi=65{,}625-42{,}133=23{,}492$ lb. ($10\frac{1}{2}$ tons), hence $\varphi=.56f$ in this experiment.

It is desirable to supplement this result by a further analysis of experiments on other sorts of wrought iron. The experiments carried out by Mr. Barlow, Mr. Clarke, and Mr. Fairbairn afford sufficient data for the purpose.

Three bars of wrought iron, 1 in. square and 16 in. long, were fixed at 12 in. bearing, and tested by a gradually increasing bending stress applied at the centre. In each of the experiments the bar slipped between the supports without positive rupture under a mean stress of 2.33 tons. The apparent strain per square inch on the extreme fibres at that time would be:

$$F=\frac{2.33 \text{ tons} \times 12 \text{ in. span} \times 6}{4 \times 1 \text{ in.}^3}=42 \text{ tons per square in.}$$

Now, the ascertained direct tensile strength of the bar was 25 tons per square inch. Hence, $\varphi=42-25=17$ tons$=.68f$.

In another series of experiments, four bars $1\frac{1}{2}$ in. square and 4 ft. 6 in. long were heated to a dull red heat in a furnace, and, whilst in that state, two of the bars were bent to a camber of 3 in. by wooden mallets, and allowed to cool with the other pair of bars which had not been touched after heating. When cool, the bent bars were straightened and tested by a weight applied at the centre of 3 ft. supports, the maximum load carried being 2.325 tons. The other pair of bars were tested in a similar manner, and gave a mean result of 2.1 tons. We have therefore for the bent bars,

$$F=\frac{2.325 \text{ tons} \times 36 \text{ in. span} \times 6}{4 \times 1.5^3 \text{ in.}}=37 \text{ tons per square in.}$$

and for the original bars we have:

$$F=\frac{2.1 \text{ tons} \times 36 \text{ in. span} \times 6}{4 \times 1.5^3 \text{ in.}}=34 \text{ tons per square in.}$$

The direct resistance to tension of the metal being 23 tons per square inch, the value of φ in the first and second set of experiments respectively would be as follows:

$$\varphi=37-23=14 \text{ tons per square inch}=.61f$$
$$\varphi=34-23=11 \quad,, \quad ,, \quad =.48f.$$

Several beams built up of a pair of angle irons and a single flange plate were experimented upon by Mr. Fairbairn. One section made up of L irons $2\frac{1}{2} \times 2\frac{1}{2} \times \frac{1}{4}$, and a plate $5 \times \frac{1}{4}$, was tested by weights applied at the centre of 27 in. bearing. With 10,423 lb., the deflection amounted to 2.9 in., and with 10,759 lb. the beam sank.

Now the moment of resistance of the cross section, calculated in the same manner as that of the piece of T iron already illustrated, being $M=.91$, the maximum apparent strain on the outer fibres of the vertical webs of the L iron would be

$$F=\frac{27 \text{ in.} \times 10{,}759 \text{ lb.}}{4 \times .91}=80{,}000 \text{ lb.}$$

The average tensile strength of the iron used in the construction of the beam being 50,000 lb., the value of φ in this instance will be 30,000 lb.; hence $\varphi=.6f$. Other experiments of a similar nature, made at the same time, gave rather lower results, the mean value of φ being about $.57f$.

Mr. Barlow made a series of experiments on the *elastic* resistance of bars varying in section from 2 inches square to 3 in. by $1\frac{1}{2}$ in., with the following mean results, the weights being in all instances applied at the centre of bearings 33 in. apart:

	Size of bar.	Strength within elastic limit.
(*a*)	3 in. × $1\frac{1}{2}$ in.	$4\frac{1}{4}$ tons.
(*b*)	$2\frac{1}{2}$ in. × $1\frac{1}{2}$ in.	3 ,,
(*c*)	2 in. × 2 in.	$2\frac{1}{2}$,,

The apparent maximum strain on the fibres in the several series of experiments would, consequently, be as follows:

$$(a)\quad \frac{4\frac{1}{4}\text{ tons}\times 33\text{ in. span}\times 6}{4\times 3^2\text{ in.}\times 1\frac{1}{2}\text{ in.}}=15.6\text{ tons.}$$

$$(b)\quad \frac{3\text{ tons}\times 33\text{ in. span}\times 6}{4\times 2\frac{1}{2}^2\text{ in.}\times 1\frac{1}{2}\text{ in.}}=15.8\text{ tons.}$$

$$(c)\quad \frac{2\frac{1}{2}\text{ tons}\times 33\text{ in. span}\times 6}{4\times 2^3\text{ in.}}=15.4\text{ tons.}$$

The mean result, therefore, of the experiments on the different sized bars gave 15.6 tons per square inch as the *elastic* resistance to tension of wrought iron when in the form of a solid beam. Now, the elastic resistance to direct tension of the same metal averaged 10 tons per square inch; hence, since the ultimate resistances would be proportional to the elastic resistances, we have

$$\varphi : f :: 15.6-10 : 10;$$

therefore in these bars $\varphi = 56f$—a result precisely identical with that arrived at in the first series of experiments analysed, namely, those on the hammered Swedish iron bars.

The several values of φ deduced from the various sets of experiments arranged consecutively are as follows:

$$\varphi = .56f;\ .68f;\ .48f;\ .61f;\ .57f;\ \text{and } .56f.$$

The mean value, therefore, will be $.57f$; or, since the same laws which govern the value of φ in different cross sections operate alike in wrought or cast-iron beams, we may safely give the following values of φ for the cross sections enumerated below:

Rectangular cross section . . . $\varphi=\frac{9}{16}f$
Round . . . . $\varphi=\frac{11}{16}f$
Square (in direction of diagonal) . $\varphi=\frac{13}{16}f$.

The values of φ for other forms of cross sections will generally be less than these tabulated results. Thus, for a girder-section, the equivalent value of φ may be taken as the fraction $\frac{\text{thickness of web}}{\text{width of flange}}$ of the amount obtained in a solid rectangular cross section. That this simple assumption is sufficiently near the truth for practical purposes is well shown by the following application of the method to a piece of **H** iron of the section shown in Fig. 5, the breaking weight of which, at 48 in. bearing, was ascertained by Mr. Fairbairn to be 6¼ tons.

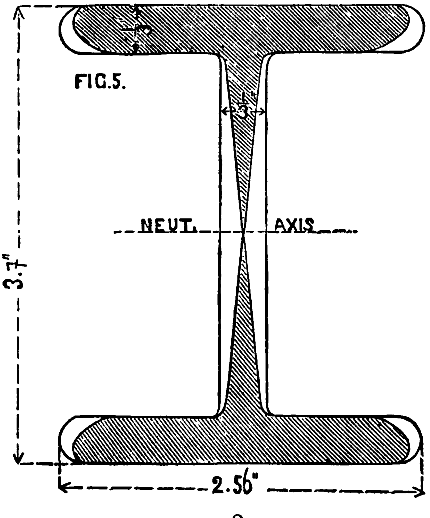


FIG.5.

Now the tensile strength of the metal being 22 tons per square inch, the value of φ on the preceding hypothesis would—since the thickness is $\frac{1}{3}$ in. and the width of flange 2.86 in.—be

$$\varphi=\frac{22 \times \frac{1}{3}\text{ in.} \times \frac{9}{16}}{2.86\text{ in.}}=1.4\text{ tons};$$

hence $F=22+1.4=23.4$ tons.

Again, the calculated moment of resistance of the cross section being $M=3.24$, the breaking weight at the centre of 48 in. bearing would be:

$$W=\frac{4 \times 23.4\text{ tons} \times 32.4}{48\text{ in.}}=6.3\text{ tons};$$

a result practically identical with that arrived at by direct experiment.

It is hardly necessary to illustrate further the application of the results deduced from the preceding experiments on wrought iron to practical cases, since the method will be precisely identical with that already exhibited when treating on cast-iron beams. We may, however, refer briefly to the piece of T iron 3 in. × 3 in. × $\frac{1}{2}$ in., which was selected to illustrate the process of determining the moment of resistance of an unsymmetrical form of cross section, the value of which in that instance we found to be $M=1.08$.

Now, suppose this T iron was an integral part of a structure, the ruling strain on which was 5 tons per square inch, and that it was required to know what weight might be brought upon it at the centre of a span of 60 in., the solution of the problem would be as follows:

Taking the value of φ as obtained in rectangular cross sections, we have:

$$F=f+\varphi=5+\frac{9 \times 5}{16}=8\text{ tons}$$

per square inch nearly, and that will be the mathematical value of the strain on extreme fibres. Again, W being the required weight, we have:

$$\frac{\text{W} \times 60\,\text{in. span}}{4} = 8 \text{ tons} \times 1.08\,;$$

hence W = .58 ton.

Now Mr. Fairbairn experimented on various sections of **T** iron, and deduced the empirical rule that the strength of a piece of **T** iron is similar to that of a flanged girder of which the depth is half that of the **T**, and the area of flange equal to that of the vertical web. In this instance, then, the equivalent area of flange would be 1¼ square inches, and the effective depth 1½ in.; hence, with a strain of 5 tons per square inch, the weight at the centre of the 60 in. span would be

$$\text{W} = \frac{4 \times 5 \text{ tons} \times 1\tfrac{1}{2}\,\text{in.} \times 1\tfrac{1}{4}\,\text{in.}}{60\,\text{in.}} = .62 \text{ tons}\,;$$

a result rather higher than that given by the previous equation, where the proper value of φ was introduced, but still quite within the limits of variation in the strength of different specimens of even the same sort of iron. It is unnecessary, therefore, to pursue our investigations on wrought-iron beams any further, so we will proceed at once to the consideration of those constructed of steel.

Section 5.

ON THE STRENGTH OF STEEL.

The prejudice entertained by the old school of engineers against steel as a material for the construction of rails, shafts, and severely strained structures in general, was, probably, to a great extent, due to the very discordant results given by the early experiments on the strength of that metal. Now, the fact is, no material is more powerfully affected by different modes of treatment than steel. The influence of Burnetising, or other preservative measures on timber, as far as regards its strength, is perfectly insignificant compared with the effect of comparatively unimportant details in the manipulation of steel. Thus, a specimen of cast steel, highly heated and cooled in oil, bore upwards of 96 tons per square inch, whilst a similar piece cooled in water failed with 40 tons, and another piece, cooled slowly in ashes, with 54 tons per square inch.

Increased hardness brings with it, generally, the advantage of increased resistance to a *steady* tensile strain, but only at the expense of a deterioration in the power of resisting a blow. We have before explained that the latter resistance will be represented by the *work done* in fracturing the bar, which is, approximately, the breaking weight multiplied by one-half the ultimate extension. Now, referring to the preceding experiments, the work done in fracturing the hard specimen was, since the ultimate elongation amounted to 3.3 per cent., proportional to 96 tons × 3.3 per cent. = 3.16; whilst, with the same steel in a soft state, the work done was proportional to 54 tons × 7 per cent. = 3.78, thus exhibiting an increase of about 20 per cent. in its power of resisting impact, although the ultimate tensile strength of the material was 44 per cent. less than before. We shall find the same rule hold good with all descriptions of steel. Hardness increases the ultimate resistance to tension, but diminishes its extensibility, and, consequently, its ability of being twisted and bent about without fracture or material injury to the metal.

In the manufacture of steel rails the desideratum is to attain the happy mean; they can hardly be too hard as far as abrasion from ordinary traffic, or planing away of the surface by skidding wheels, are concerned; and they can hardly be too soft, if we consider only the liability to fracture by rough handling, even in loading or unloading, or facility in laying the permanent way with the practical details of cutting and drilling the rails in cross-over roads and sidings.

A very extensive and valuable series of experiments on the strength of steel plates was carried out by Mr. Kirkaldy; the mean tensile strength per square inch of about 80 specimens of different manufacture was found to be 85,966 lb. (38½ tons), the mean resistance of each set of plates ranging from 71,792 lb. to 106,110 lb. per square inch. The ultimate elongation averaged 8 per cent. only.

A box girder constructed of puddled steel plates was tested by Mr. Fairbairn, and at first sight, it must be confessed, results appeared confirmatory of the prejudice very generally entertained at that time against the adoption of steel in such structures. The girder tested was 37 ft. 6 in. span, and 2 ft. 3 in. deep; the web plates, $\frac{1}{10}$th of an inch thick, were riveted by four angle irons, 1½ in. × 1½ in. × ¼ in., to the bottom flange, 18 in. × ¼ in. thick, and in the same manner to the top flange, which was $\frac{5}{16}$th of an inch thick. The unexpected result of the tests applied was that failure occurred by the rupture of the bottom flange under a load which brought a calculated strain of 22 tons per square inch only on that member—an amount hardly exceeding that sustained by girders constructed of ordinary bridge iron.

With the view of corroborating this calculated tensile breaking strain, some pieces of metal were cut from the beam, and tested by direct tension, when it was found that the highest resistance offered was but 27.7 tons per square inch, and that some portions failed with 25.9 tons only. Now, previous experiments on the puddled steel plates gave an average of 41.8 tons as the ultimate tensile strength per square inch; so that the metal actually used in the construction of the beam exhibited a deficiency in strength of about 37 per cent. This, of course, at once clears up the apparent anomaly of the steel girder being little if any stronger than a first-rate iron one of similar dimensions. Had the metal been of the quality stipulated, the girder would have borne more than half as much again load. The inferior quality of the *metal*—for it could not fairly be styled *steel*—used was also evidenced by its ultimate elongation, which averaged but 4½ per cent., whilst in the steel originally tested it amounted to 11 per cent. Even the latter amount is rather low for the material, since other specimens of puddled steel, tested by Kirkaldy, elongated from 19 to 35.3 per cent. under the several ultimate tensile strains. On the other hand, some rivet steel which failed with precisely the same strain as the original plates of Mr. Fairbairn's girder, attained some 2 per cent. less elongation, and some soft tool steel of Mr. Bessemer's, with nearly 9 tons per square inch greater strain, was another 2 per cent. below the latter in the rate of elongation.

The tensile strength of Bessemer steel in plates generally averages about double that of good iron plates; in fact, the *elastic* resistance of the former material, that is, the point at which the increments of elongation cease to be regular, is practically identical with the *ultimate* resistance of the latter. Thus, in a number of pieces of Bessemer steel plate, ⅝ in. thick and 1½ in. wide, the ultimate tensile strength ranged from 43 tons to 46 tons per square inch, the mean being 44½ tons, and the elastic resistance from 20⅓ tons to 20½ tons, the average being 22 tons per square inch. Another set of steel plates, tested by Samuda, Brothers, broke under an average load of 43½ tons per square inch, and the mean of twenty-eight experiments made at Chatham on steel plates for the Bellerophon gave 39.8 tons per square inch as the ultimate resistance to tension.

The Chief Constructor of the Navy, in his very valuable work on "Ship-building in Iron and Steel," gives the results of some experiments on batches of puddled and Bessemer steel, sent in by different makers, to undergo a tensile strain of 33 tons per square inch lengthways, and 30 tons across the grain. In the puddled plates the variation in the strength lengthways was from 26.2 to 36.7 tons, and crossways from 23.8 to 33 tons per square inch. In the case of the Bessemer steel plates, the range was from 35 to 49 tons per square inch; but these specimens were by no means so uniform as usual. The decrease in the elongation of the plates as the strength increases, Mr. Reed observes, was very noticeable; in fact, the rate of elongation varied inversely very nearly as the cube of the ultimate resistance per square inch.

There is little, if any, difference between the tensile strength of Bessemer steel plates and rails of the same material, nor does the section of the rail materially affect the resistance per unit of area, as will be seen by the following experiments on 86 lb. and 68 lb. double-headed rails, manufactured by Sir J, Brown and Co. and Messrs. Cammell and Co.:

86 *lb. Rail.*

Tensile strength of steel in web (four experiments).

$50\frac{1}{2}$ tons, $45\frac{1}{2}$ tons, $33\frac{1}{2}$ tons, $45\frac{1}{2}$ tons. Mean $43\frac{3}{4}$ tons per square inch.

Tensile strength of steel in head (three experiments).

47 tons, $44\frac{2}{3}$ tons, $42\frac{2}{3}$ tons. Mean $44\frac{7}{9}$ tons per square inch.

68 *lb. Rail.*

Tensile strength of steel in web (five experiments).

50 tons, 37 tons, 38 tons, 45 tons, 48 tons. Mean $43\frac{3}{5}$ tons per square inch. The elongation of metal ranged from $6\frac{1}{4}$ per cent. in the stronger, and consequently harder, specimens, to double that amount in those attaining the lower tensile resistances only. The mean result might be taken as 10 per cent.

We have been favoured by A. Longsdon, Esq., with the results of some experiments made by Mr. Kirkaldy to ascertain the extension, set, and ultimate resistance to tension of steel in various forms manufactured by Krupp, at the Essen works. Cast-steel boiler plates $\frac{3}{8}$ in. and $\frac{1}{2}$ in. thick were tested, with the following results:

Thickness of plate.	Elastic resistance.	Ultimate resistance.
3/8	52,000 lb. per sq. in.	83,183 lb. per sq. in.
"	do. do.	82,070 " do.
"	do. do.	81,655 " do.
"	do. do.	80,127 " do.

Mean ultimate resistance to tension, 81,759 lb. (36¼ tons) per square inch. Mean ultimate elongation, 21.2 per cent.

Thickness of plate.	Elastic resistance.	Ultimate resistance.
1/2	41,000 lb. per sq. in.	66,784 lb. per sq. in.
"	do. do.	66,525 " do.
"	42,000 " do.	66,730 " do.
"	44,000 " do.	66,086 " do.

Mean ultimate resistance to tension, 66,531 lb. (29¾ tons) per square inch. Mean ultimate elongation, 21.2 per cent.

With the evidence already advanced in these papers before us, it would be a work of supererogation to prove that almost anything might be done with plates possessing the attributes shown by the preceding experiments to belong to those manufactured at Essen. We have seen specimens folded cold, like so much paper, and with just as little apparent injury to the material; indeed it might have been predicated from the observed rate of elongation under tensile strains that such would be the case. The most tractable specimens of Lowmoor or Bowling plates are quite distanced by these cast-steel plates. Thus, the mean ultimate resistance of the latter, as deduced from the preceding experiments, was 33 tons per square inch, and the mean ultimate elongation, 22.8

per cent.; and we know that the work done in rupturing a plate would be proportional to the product of those figures. But the average results given by the best makes of iron plates, tested by Kirkaldy, was 25⅓ tons per square inch, ultimate tension, with 14.15 per cent. elongation, and the mean of all the experiments on different descriptions of plate gave 21¾ tons per square inch as the ultimate resistance to tension, and 7 per cent. as the ultimate elongation. Hence, since the "work done" in rupturing the respective plates or the amount of "fatigue" they are capable of sustaining, proportional to the product of the stress into the elongation, the values of the several averages will in terms of the average iron plate, be 2.35 for the best make of iron, and 4.9 for the cast steel plates.

The following are the results of experiments on two specimens of steel cut out of a Krupp tyre:

Elastic resistance.	Ultimate resistance.	Elongation.
71,000 lbs. per sq. in.	114,480 lb. per sq. in.	22.8 per cent.
70,000 ,,	113,880 ,,	23.4 ,,

It appears, then, that more duty can be got out of the material in the more solid section than when it is in the form of plates, since with practically the same amount of elongation the tensile resistance is increased to 51 tons per square inch. We have already found the same conditions obtain in bar iron.

The conclusions we have already drawn from observations on the behaviour of wrought iron under compressive strains apply with equal force to steel. In both materials the area of the parts in compression increases, *pari passu*, with the imposed strain; consequently, in a short column, or, what amounts to much the same thing, in a short solid bar fixed laterally and subjected to transverse strain, it would be an unnecessary labour to enter into any calculation of the ultimate compressive strains. This will be clearly evidenced by the following experiments on four short columns of hammered cast steel, 1¼ in diameter:

Mean of four Experiments.

Elastic resistance.	With 35 per cent. depression.
19¼ tons per sq. in.	58 tons per. sq. in.

In one specimen with 36 per cent. depression the stress sustained amounted to 95 tons per square inch.

The elastic resistance to tension of this steel was identical with its resistance to compression, but the ultimate elongation of 19.2 per cent. was attained under a rupturing strain of 32¼ tons per square inch. Hence it appears that the change of length in the specimens under a given strain per square inch was about the same in amount, whether it was excited by a pulling or thrusting stress, but whilst under a tensile strain the increment of length was limited to 19.2 per cent., the decrement, under a compressive strain of *treble* the ultimate tensile resistance, amounted to but *double* the ultimate elongation, and the limit of resistance to compression was not even then attained.

We are indebted to A. Longsdon, Esq., for the following tabular results of the very interesting and unique series of experiments, carried out by Mr. Kirkaldy at his Testing-works in Southwark, on a piece of the broken cast-steel crank shaft of the Peninsular and Oriental Company's screw steamship Jeddo, manufactured by Krupp, at Essen.

Referring to Figs. 6 and 7, it will be seen that a length of the broken end of the shaft was sliced up into specimens, which were subjected to pulling, thrusting, bending, and twisting stresses. A more severe test of the quality and uniformity of the steel could not have been devised, and the admirable manner in which the several stresses were sustained could not have been surpassed. The experiments afford ample evidence that the shaft could not have broken under any exigency of fair working. We understand that it was due to the culpable negligence of the engineers, who first allowed the bearings to heat, and then attempted to mend matters by throwing water upon them.

TABLE A.—Resistance to Extension, Set, and Rupture under Tensile or Pulling Stress.

The relative Position of the Specimens is represented in Fig. 7, the Shape, &c., of Specimens in Fig. 6.

Test	ORIGINAL.	STRESS IN POUNDS PER SQUARE INCH.—LENGTH OF SPECIMENS FOR EXTENSION 8.5 INCHES.—EXTENSION AND SET, INCH.												ULTIMATE STRESS.		FRACTURED.		STRESS.	EXTENSION.	
No.	Dia.	40,000	48,000	56,000	64,000	72,000	80,000	84,000	88,000	92,000	96,000	100,000	104,000	Total.	Per Sq. Inch of Original Area.	Diameter	Difference of Area Per Cent.	Per Sq. Inch of Fractured Area.	Total.	Per Cent.
	inch.													lb.	lb.	in.		lb.	in.	
743	1.25	.043	.069 .013	.128 .049	.188 .091	.250 .148	.330 .218	.377 .253	.435 .312	.505 .364	.590 .451	.715 .568	.913 .740	128,420	104,642	— Broke at the head B. —			0.78	9.2
739	1.25	.047	.071 .017	.130 .062	.190 .111	.268 .174	.364 .262	.417 .310	.482 .370	.570 .453	.700 .574	.895 .770	...	126,940	103,447	1.11	21.1	131,177	7.36	16.0
741	1.25	.046	.068 .014	.132 .061	.194 .116	.276 .179	.371 .266	.426 .316	.494 .374	.586 .458	.709 .572	.896 .764	...	126,680	103,235	1.12	19.7	128,583	1.26	14.8
755	1.25	.044	.097 .041	.158 .091	.230 .152	.320 .230	.432 .333	.508 .398	.600 .490	.730 .612	.924 .810	...	...	121,080	98,671	1.13	11.3	120,729	1.16	13.6
760	0.75	.041	.100 .050	.160 .092	.233 .158	.327 .244	.450 .360	.430 .435	.648 .552	.852 .733	.136	...	...	42,576	96,391	0.66	22.5	124,455	1.29	15.2
753	1.25	...	...	...	...	...	...	...	...	...	...	...	...	117,340	95,624	1.12	19.7	119,102	1.16	13.6
761	0.75	.032	.091 .043	.153 .093	.222 .153	.320 .237	.455 .366	.540 .449	.660 .568	.907 .805	...	...	...	41,748	94,516	0.66	22.5	122,034	1.23	14.4
747	1.25	.041	.095 .044	.164 .095	.241 .165	.334 .247	.465 .368	.560 .463	.710 .603	.967 .869	...	...	...	115,096	93,795	1.11	21.1	118,939	1.03	12.1
757	1.25	...	...	...	...	...	...	...	...	...	...	...	...	114,880	93,619	1.09	24.0	123,103	1.26	14.8
759	0.75	.044 .010	.108 .061	.168 .110	.245 .180	.340 .267	.470 .391	.573 .483	.715 .618	.980 .890	...	...	...	41,004	92,830	0.65	24.9	123,580	1.17	13.8
762	0.75	.046 .011	.107 .060	.165 .105	.243 .174	.340 .262	.482 .400	.582 .495	.730 .631	1.03 0.95	...	...	...	40,848	92,479	0.65	24.9	123,110	1.20	14.1
745	1.25	...	...	...	...	...	...	...	...	...	...	...	...	112,210	91,442	— Broke at the head B. —			0.87	10.2
751	1.25	.064 .023	.125 .073	.192 .135	.285 .210	.418 .330	.600 .505	.772 .670	...	...	...	...	...	107,896	87,928	1.08	25.3	117,738	1.39	16.4
735	1.25	.065 .024	.130 .075	.203 .140	.304 .230	.448 .366	.700 .602	.103 094	...	...	...	...	...	103,656	84,472	1.07	26.7	115,276	1.47	17.3
														Mean	95,221		22.6	122,322	Mean	14.0
737	1.25	...	.100 .041	.163 .093	.235 .153	...	...	...	...	...	...	...	...	88,240	71,909	— Broke at the head B. — Previously cracked in connexion with the breakage.			0.32	3.8
749	1.25	... .046 .005	... .097 .044	... .160 .093	... .234 .157	... .331 .244	... .465 .370	... .574 .474	...	...	...	...	...	86,820	70,752	— Broke at the head B. —			0.17	2.0
														Mean	92,235		22.6	122,322	1.26	12.6

TABLE B.—Resistance to Torsion, Set, and Rupture under a Twisting Stress. Length for Torsion=Two Diameters.

The relative Position of the Specimens is represented in Fig. 7, the Shape, &c., of Specimens in Fig. 6.

Test	ORIGINAL.		LENGTH OF LEVER, 12 INCHES.—STRESS IN POUNDS ON EACH END.—TORSION, ONE TURN=1.000.													ULTIMATE STRESS.		Ultimate	REMARKS.
No.	Dia.	Area.	.800	1000	1200	1400	1600	1800	2000	2200	2400	2600	2800	2900	3000	On 12 in. lever. Total.	Per square in.	Torsion.	
																lb.	lb.	Turn.	
742	1.25	1.2271	.002	.005	.010	.018	.030	.040	.055	.080	.096	.140	.221	.282	3.71	3006	2449	0.386 0.463	1st end, fractured. 2nd end, nearly do.
		Set			.006	.015	.023	.035	.049										
738	1.25	1.2271	.002	.003	.008	.015	.025	.036	.050	.078	.112	.145	.254	.345	...	2928	2386	0.376 0.398	1st end, fractured. 2nd end, nearly do.
		Set			.004	.012	.020	.030	.046										
746	1.25	1 2271	.001	.004	.008	.017	.029	.046	.067	.098	.145	.214	.345	...	...	2891	2356	0.432 0.468	1st end, fractured. 2nd end, nearly do.
		Set			.005	0.14	.025	.041	.061										
750	1.25	1.2271	.001	.003	.015	.029	.041	.064	.100	.151	.250	.472	...	...	...	2628	2142	0.495 0.508	1st end, fractured. 2nd end, nearly do.
		Set			.013	.025	.037	.060	.094										
		Mean	.001	.004	.010	.020	.031	.046	.068	.102	.151	...	...	...	...	Mean	2333	0.441	
		Mean			.007	.016	.026	.041	.062										

TABLE C.—Resistance, Deflection, Set, and Rupture, under a Transverse or Bending Stress. Distance, Ten Inches.

The relative Position of the Specimens is represented in Fig. 7, the Shape, &c., of Specimens in Fig. 6.

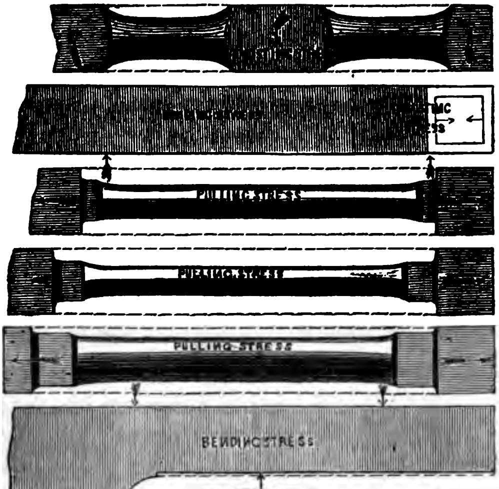
Test No.	DIMENSIONS.	B D²	STRESS IN POUNDS PER B D².—DEFLECTION AND SETS, INCHES.														ULTIMATE.			VALUE OF S.
			1000	2000	3000	4000	5000	6000	7000	8000	9000	10,000	10,500	11,000	11,500	12,000	Deflection.	Stress.	$\frac{\text{Stress.}}{\text{B D}^2}$	$S = \frac{L W}{4 B D}$
	Breadth. Depth.																			
740	1.37 × 1.76	4.2437	.019	.029	.038	.048	.061	.104	.170	.253	.376	.518	.603	.711	.853	1.09	1.16	51,668	12,175	30,438
		Set						.030	.087	.171										
744	do. × do.	4.2437	.022	.036	.049	.057	.081	.133	.198	.280	.392	.529	.610	.729	.889	...	1.26	50,792	11,969	29,922
		Set					.017	.062	.116	.187										
754	do. × do.	4.2437	.023	.031	.041	.050	.082	.146	.225	.322	.442	.660	.752	.913	1.17	...	1.46	50,288	11,850	29,625
		Set					.016	.074	.134	.222										
736	do. × do.	4.2437	.017	.025	.039	.050	.090	.166	.252	.360	.520	.762	.961	1.45	...	...	1.63	47,460	11,183	27,958
		Set					.023	.092	.156	.255										
752	do. × do.	4.2437	.021	.031	.046	.054	.110	.193	.288	.407	.578	.850	1.05	1.61	...	...	1.73	47,312	11,149	27,872
		Set					.047	.120	.201	.310										
756	do. × do.	4.2437	.017	.031	.040	.050	.106	.190	.297	.418	.602	.892	1.10	1,66	...	...	1.77	47,028	11,082	27,705
		Set					.042	.112	.205	.315										
748	1.37 × 1.71	4.0460	.015	.027	.035	.050	.102	.194	.310	.460	.660	1.02	1.41	...	...	...	1.53	42,872	10,702	26,755
		Set					.040	.118	.217	.353										
758	1.37 × 1.76	4.2437	.015	.027	.037	.075	.153	.250	.373	.545	.799	1.27	...	...	...	...	1.41	44,376	10,457	26,142
		Set				.022	.085	.177	.282	.448										
		Mean	.019	.029	.041	.054	.098	.172	.264	.381	.546	.810	.927	1.179	...	...	1.49	47,725	11,321	28,302
		Mean					.034	.098	.175	.283										

TABLE D.—Comparative Summary of the Results of Experiments on Pieces cut out of one of the Crank Spaces of Two New Wrought-Iron Shafts, made at Glasgow, for the Screw Steamship Jeddo, and on Pieces cut out of the Broken Crank of Cast Steel Shaft, made at Essen, Prussia

IRON SHAFTS.					CAST STEEL SHAFT.										
PULLING STRESS.					PULLING STRESS.					BENDING STRESS.		THRUSTING STRESS.		TWISTING STRESS.	
Ultimate Stress per Square Inch of original Area.		Contraction of Area at Fracture.	Ultimate Stress per Sq. Inch of Fractured Area.	Ultimate Permanent Extension.	Ultimate Stress per Square Inch of original Area.		Contraction of Area at Fracture.	Ultimate Stress per Sq. Inch of Fractured Area.	Ultimate Permanent Extension.	Ultimate Deflection.	Value of S. $S=\frac{L\ W}{4B\ D^2}$	Ultimate Permanent Depression with 220,000 lb. per Square Inch.		Ultimate Stress. lb. per Sq. In. on 12 in. Lever.	Ultimate Torsion.
lb.	lb.	per ct.	lb.	per ct.	lb.	lb.	per ct.	lb.	per ct.	in.		in.	per ct.	lb.	One turn=1.000
49,671					104,642					1.16	30,438	.394	31.6		
47,591					103,447									2,449	0.386
49,167	47,582	20.7	59,003	21.9	103,235					1.26	29,922	.428	34.3		0.463
45,234					98,671										
47,459					96,391					1.46	29,625	.433	34.7	2,386	0.376
46,450					95,624										0.398
					94,516	95,221	22.6	122,322	14.0	1.63	27,958				
					93,795										
					93,619					1.73	27,872	.472	37.8	2,356	0.432
					92,830										0.468
44,561					92,479					1.77	27,705				
44,311					91,442									2,142	0.495
43,420	43,759	23.1	56,910	20.5	87,928					1.53	26,755	.486	88.8		0.508
43,420					84,472										
43,420					71,909	71,330	...	...	2.9	1.41	26,142	.488	39.1		
43,420					70,752										
Mean	45,670	21.9	57,956	21.2	Mean	92,235	22.6	122,322	12.6	1.49	28,302	.450	36.1	2,333	0.441

Note.—The Experiments on the Iron Shafts were made by Mr. Kirkaldy, and the above Results are from his published "Experiments on Wrought Iron and Steel," pages 116, 117, 134, 135.

Fig. 6.


MC
SS
PULLING STRESS
PULLING STRESS
PULLING STRESS
BENDING STRESS

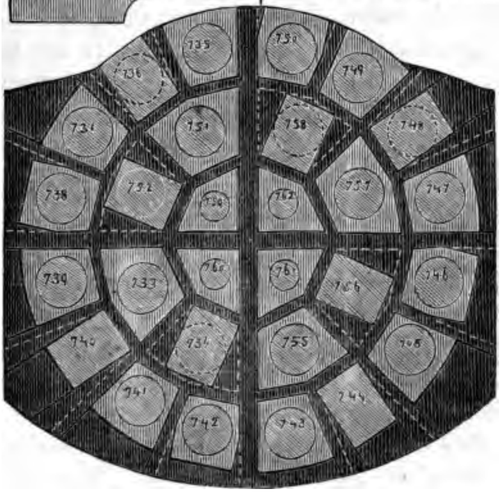


Fig. 7.

It appears from the preceding experiments that the average ultimate tensile resistance of the steel in the shaft, excluding the specimens previously injured by the fracture, was 95,221 lb. (42½ tons) per square inch—a resistance, as we have already shown, practically the same as that offered by ordinary double headed Bessemer steel rails. But the rate of elongation of the steel in the shaft averaged 12.6 per cent.; and if we exclude, as we may fairly do, the specimens which broke at the head before the full strain came upon the metal, the mean elongation would be 14.7 per cent., whilst, it may be remembered, for steel rails we found the rate to be 9⅜ per cent.; hence, with the same ultimate strength in both instances, the steel in the shaft had the advantage of additional toughness, and, consequently, greater capability of resisting rough usage. The *elastic* resistance to tension, as indicated by these experiments, averaged 19 tons per square inch, which is a rather low fraction of its ultimate resistance, as compared with that obtained in other experiments on various makes of steel; but the fact is, the exact point at which the increments of extension cease to become regular is so difficult to determine, that, even when the appliances at hand are of the same refined nature as those Mr. Kirkaldy has at his disposal, the results of such delicate tests are not to be implicitly relied upon.

The experiments on the resistance to compression support the conclusions, already drawn from previous ones, as to the practically unlimited extent to which weight may be piled upon a short column of steel without effecting its positive destruction. Thus, in these experiments the enormous stress of 98 tons per square inch merely entailed a depression of 36.1 per cent. in the specimens, and no doubt, as the area resisting the pressure would have increased with the depression, a very much greater load might have been brought to bear had it been considered necessary.

Having sufficient evidence before us as to the *direct* tensile resistance of steel of various makes and in various forms, we may proceed with the investigation necessary to determine the value of the same resistance when excited by transverse strain; or, in other words, the *apparent* tensile resistance of the extreme fibres in a solid steel beam. We could not obtain more satisfactory data for this purpose than that afforded by the experiments on the broken steel shaft, since we have there the *direct* tensile resistance, and also the necessary information for deducing the *apparent* tensile resistance, of practically the same specimen of metal.

Thus, in Table C we have the several values of

$$S=\frac{W\,L}{4\,b\,d^2},$$

and since, as we have already shown, for a rectangular cross section the *apparent* strain per square inch on the extreme fibres will be

$$F=\frac{6\,W\,L}{4\,b\,d^2},$$

we can at once deduce the value of that strain from the several experiments by simply multiplying the tabular value of S by 6.

Now in Table C we find the mean value of S to be 28,302 lb.; the highest value (No. 740) 30,438 lb. and the lowest (No. 758) 26,142 lb.; and in Table A we find the mean value of the tensile resistance per square inch, f, to be 95,221 lb. Referring now to Fig. 6, we find the specimens tested by direct tension adjacent to the one offering the highest transverse resistance (No. 740) to be Nos. 739, 753, and 741, the ultimate strengths of which are shown, in Table A, to have been 103,447, 95,624, and 103,225 lb. per square inch; we may, therefore, take the average of those amounts = 100,768 lb. per square inch as the true value of f in the specimen (No. 740).

Proceeding in the same manner with the weakest beam (No. 758), we find the nearest specimens to it to be Nos. 749, 751, 762, and 757, with resistances respectively of 70,752, 87,928, 92,479, and 93,619 lb. per square inch. The mean value of f in this instance will, consequently, be 86,194 lb. per square inch. Tabulating the preceding results, we arrive at the following values of F, f, and φ, for the average, strongest, and weakest specimen tested by transverse stress.

	F	f	$\varphi = F - f$	
Average	$6 \times 28,302 = 169,812$	95,221	74,591	$\varphi = .785 f$
Strongest ...	$6 \times 30,438 = 182,628$	100,768	81,460	$\varphi = .807 f$
Weakest	$6 \times 26,142 = 156,852$	86,194	70,658	$\varphi = .820 f$
Mean	...	...	...	$\varphi = .8 f$

We may check this mean value of φ by dealing in the preceding manner with the highest and lowest results attained by any specimen under pulling and bending stresses, irrespective of their relative positions in the cross section of the shaft. The results are as follows:

	F	f	$\phi=F-f$	
Strongest …………	182,628	104,642	77,986	$\varphi=.745f$
Weakest ……………	156,852	84,472	72,380	$\varphi=.855f$
Mean ………………	…	…	…	$\varphi=.8f$

The average value of φ in both instances being $.8f$, we may safely adopt that amount for cast steel similar to that employed in the manufacture of the broken shaft.

It is necessary now to supplement the foregoing experiments by some others on different makes of steel.

In a paper read before the British Association, Mr. Fairbairn recorded the results of forty-five experiments on the transverse strength of steel of different makes. The mean value of F at the *elastic* limit was found to be 41 tons, whilst the value of f at the same limit averaged 24 tons per square inch, hence $\varphi=.708\,f$ in those experiments.

On another occasion four bars of soft chisel steel ¾ in. square were tested by direct tension, and by weights applied at the centre of 12 in. bearings; the mean breaking weights under the respective conditions proved to be 26 tons and 1.85 tons.

Now, W being the breaking weight in the centre of the span L, and w the direct tensile resistance of the same square bar, the side of which $=d$; we have

$$f=\frac{w}{d^2}, \text{ and } F=\frac{6\,W\,L}{4d^3}.$$

But $\varphi=F-f$, hence the value of φ in terms of f will be

$$\varphi=\frac{3\,W\,L}{2wd}-1.$$

Substituting the values given by the experiments, we have:

$$\varphi=\frac{3\times 1.85\text{ tons}\times 12\text{ in.}}{2\times 26\text{ tons}\times \frac{3}{4}\text{ in.}}-1\text{ hence }\varphi=.7\,f.$$

The following experiments were made by Mr. Kirkaldy:

Four bars of hammered hematite steel, 2 in. square, were planed down to 1.75 in. square, and tested by bending stress applied at the centre of 25 in. bearings, the strongest bar sustained 16,720 lb., the weakest 15,784 lb., and the mean result of the four experiments gave 16,474 lb. as the ultimate transverse strength of the bars.

Again, four bars of the same steel forged to 1½ in. diameter, and turned down to 1¼ in., were tested by direct tension, when the maximum resistance offered was found to be 81,108 lb. per square inch. The minimum, 69,145 lb., and the average, 72,280 lb. per square inch.

$$\text{Now since } F=\frac{6\text{ W L}}{4\,d^3},$$

we have, by substitution,

$$F=\frac{6\text{ W }25''}{4\times 1.75^3},\text{ hence } F=7\text{ W}.$$

Tabulating the results, as before, we obtain the following values of ϕ:

	F	f	$\phi=f-P$	
Average	115,318	72,280	43,038	$\phi=.595$
Strongest	117,040	81,108	35,932	$\varphi=.432$
Weakest	110,488	69,145	41,343	$\phi=.528$
Mean	...	...	...	$\phi=.54$

Again, the highest value of φ given by any two experiments would be as follows:

	F	f	$\phi=F-f$	
	115,318	69,145	46,173	$\phi=.66$

We may, therefore, consider the true value of φ in these specimens of steel to be about $.6f$; and, since in other experiments we have deduced the values $.8f$, and $.7f$, we may assume that its range will be comprised within those limits, and that for the several forms of cross sections enumerated, the values will vary as follows, according to the nature of the steel:

Rectangular cross section . . $\varphi=.6$ to $.8$
Circular . . . . $\varphi=.7$ to $.9$
Square (strained diagonally) . . $\varphi=.8$ to 1.0

The preceding deductions, coupled with those previously arrived at when treating on the ultimate direct tensile resistance of various specimens of steel, will make it evident to any one who has followed the investigation thus far, that the probable transverse strength of a given section of steel rail will be comprised within rather wide limits. We have, however, advanced sufficient data to enable those limits to be defined. Given the actual value of F in the particular case under consideration, of course the transverse strength of the rail could be obtained immediately; but we have seen that not only does the function φ increase in value relatively to f in proportion to the hardness of the steel, but that the latter quantity also is governed to a still greater extent by the same condition. From numerous experiments on pieces of metal cut out of Bessemer steel rails, the tensile resistance, *f*, appears to range from 35 tons to about 52 tons per square inch, according as the steel is mild or strong.

From a careful general consideration of the behaviour of a very large number of rails under falling weights, as well as under steady bending stresses, considered with reference to the ascertained tensile strength of the steel in the same rails, we have arrived at the conclusion that in Bessemer steel rails exhibiting the most desirable mean of hardness and toughness, the tensile resistance of the metal ranges from 40 tons to 45 tons per square inch, in double headed rails, and from 35 tons to 40 tons per square inch in flanged rails; and we consider rails which bear very much greater or smaller bending stresses than those due to the above values of *f* to be alike undesirable.

There are several practical reasons why a rather softer metal is preferable for a flanged rail, and there is one of immense importance, arising from the necessity of punching or drilling holes in the flanges for the fang bolts, by which such rails are usually secured to the sleepers. This system involves an almost incredible diminution in the power of the rail to resist a heavy blow; in fact, it is quite within the bounds of possibility that imperfect packing of the sleepers would be sufficient to induce fracture under the pounding action of the driving wheels of a heavy locomotive, when a punched steel rail of too high a tensile resistance, with the accompanying excess of hardness, has been adopted for the permanent way.

It will be well now to recapitulate briefly the general principles developed, and data accumulated, up to the present stage of the investigation.

It has been shown that the direct tensile strength of cast iron, wrought iron, and steel, of different makes, varies from the extent of some 100 per cent. in the instance of the former metal to as much as 200 per cent. in steel. For average practical cases, as far as beams or girders are concerned, it would appear that the direct ultimate resistance to tension might be taken as 7 tons per square inch for cast iron, and treble that amount for wrought iron, and that the strength of steel again might be considered to be double that of the latter metal. These results, however, show that the customary method of proportioning a structure to a given unit strain, irrespective of the particular quality of metal proposed to be employed, should be exercised with discretion.

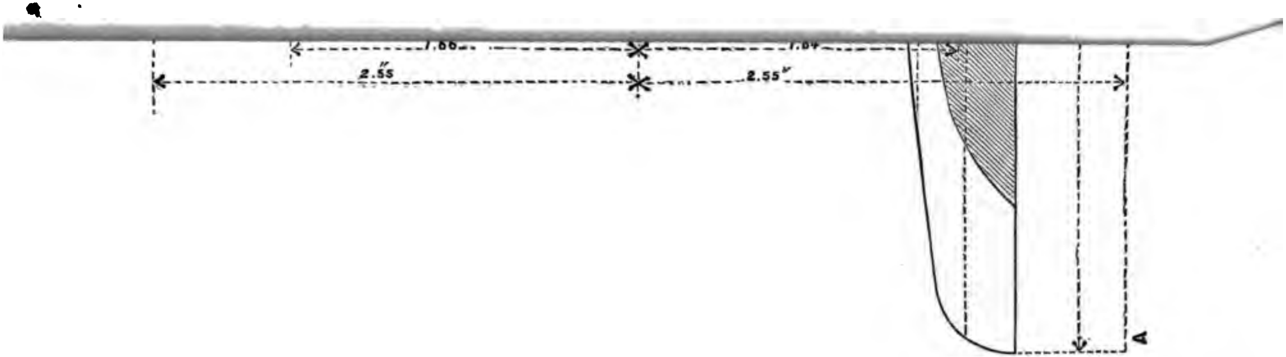
A still greater want of uniformity was exhibited in the rates of elongation, and the reduction of sectional area at the point of fracture, of different specimens of metal under the ultimate strains; involving, of course, corresponding irregularities in the amount of work done in effecting rupture in each case.

It was proved by experiment, also, that the resistance of a short column of either of the metals to direct compression was far in excess of its resistance to tension; and that, consequently, the former element might be safely neglected in estimating the probable strength of a solid beam and need not be taken into account, in any beam of symmetrical cross section, unless the compressed member were governed by the laws affecting long columns, or the buckling of thin plates, as in the instance of the top flanges of ordinary plate girders.

An analysis of numerous experiments on the transverse strength of each metal showed that in all instances and for all forms of cross section, the deduced mathematical value of the maximum strain on the extrem fibres was in excess of the ascertained direct resistanc to tension, and that, consequently, another element o strength must exist. It was ascertained that the departure from the mathematical value of the maximu strain was greatest in cast iron, where for a rectangula section the excess of strength averaged 125 per cent. and that for wrought iron the excess was about one-hal that amount, whilst steel occupied an intermediat position between the two. For other forms of cros section the value of the new element of strength appeare to vary directly as the thickness of metal in the "web portion, in terms of the mean width of the "flange portions.

A method was exhibited by which the mathematic value of the maximum strain on the extreme fibre could be ascertained geometrically; and it was show that by increasing the value of the amount so arrive at by a certain percentage, indicated by the precedin considerations, the probable transverse strength of an given section of beam might be easily estimated.

Having waded through this necessary amount of preliminary investigation, we are now in a position to predic with confidence, and with a full appreciation of th various principles involved—a very different thing t blind empiricism—the maximum resistance any steel ra would offer to a bending stress, or to any other te which might be applied to it.


1.66
1.04
2.55"
2.55"
A

Section 6.

ON THE TRANSVERSE STRENGTH OF STEEL RAILS.

We could not select a better example for exhibiting the practical application of the general principles arrived at in the foregoing investigation than the flanged steel rail, designed by John Fowler, Esq., for the Metropolitan Railway, and manufactured by the Dowlais Iron and Steel Company.

A full-sized section of this rail is exhibited in Fig. 8, by reference to which it will be seen that, in order to obtain adequate bearing surface on the cross sleepers, and room for the large fang bolts, the flange of the rail is made the unusually great width of 6⅜ in. Now, experience indicated that the heavy fang bolts, involving the large holes in the flanges, were essential to the economical maintenance of the permanent way; and since it was well known that the effect of the sudden reduction in the sectional area of the rail through the hole would be analogous to that resulting from the "nicking" of a bar, the desideratum was to obtain the *toughest* possible steel—that which would show the maximum amount of "work done" in effecting rupture—for the purpose of these rails.

The various tests which the rails have endured evidence most clearly how completely the desired result has been attained, and the remarkable uniformity of quality which characterised the different lots turned out reflects the highest credit on the management of the magnificent resources at the command of the Dowlais Company.

We will now proceed to estimate the transverse strength which it is desirable these rails should exhibit;

for, as we have previously shown, an excess of strength in this instance, involving a falling off in the toughness, would be at least as objectionable as a deficiency in the ultimate transverse resistance. Now, since the steel was required to be as tough as practicable, we might, on the data already advanced, have assumed that, if the desired result were attained, the direct tensile resistance would be as it was, about 35 tons per square inch. Again, the "flange" portions of the cross section being of considerable area relative to the "web" portion, the condition of the rail will be approximated to that of an ordinary girder; hence the difference between the apparent and the actual strain on the fibres of steel will be comparatively small. Assuming, as in previous examples, this difference to be less than that attained in a rectangular cross section in the ratio of the thickness of web to mean width of flange, and the value of the excess in a square bar of steel of the desired quality, as deduced in a previous paper, to be $\varphi = .6\,f$, we have, since the thickness of web $= \frac{11}{16}$ in., the mean width of flange $\frac{71}{16}$ in., and $f = 35$ tons per square inch,

$$\varphi = \frac{35 \text{ tons} \times .6 \times 11}{71} = 3.3 \text{ tons per square inch.}$$

Hence, the mathematical value of the strain on the extreme fibres of the rail, equivalent to 35 tons per square inch direct tensile strain, will be $F = f + \varphi = 35 + 3.3 = 38.3$ tons per square inch.

Having proceeded thus far, it is only necessary to ascertain the moment of resistance of the cross section, and to insert the preceding value of F in the equation $W = \frac{F\,M}{m}$, in order to arrive at the ultimate transverse *strength of the rail.*

Referring to Fig. 8, it will be seen that the process is identical with the one already exhibited when ascertaining the moment of resistance of the T-iron section. The first thing necessary to be determined is the position of the neutral axis; and that is done by first assuming a position for it, and setting off the line, A—B, at a distance from the neutral axis equal to that of the most remote fibres in the head of the rail. The next step is to divide the rail vertically by equidistant and parallel lines, and to square down the points of intersection of these lines with the contour of the rail, to the line A—B for the points below the neutral axis, and to the line C—D for those above that line. To complete the construction, it is only necessary to draw from the points so obtained on the bases A—B, and C—D, lines radiating to the neutral axis, and to connect the several points of intersections of these lines with the horizontal ones previously set off. Then, if the areas of the figures so bounded (shaded on the diagram) be equal above and below the neutral axis, the position of the latter point has been correctly assumed; if not, a fresh trial will be necessary, and with a little judgment there will be no difficulty in fixing the new position of the neutral axis correctly.

Proceeding in this manner, we find the position of the neutral axis to be 2.55 in. from the head of the rail; the effective area of metal above and below the neutral axis, as shaded on the diagram, being 2.49 square inches.

It is necessary now to determine the centres of pressure of the forces acting on either side of the neutral axis. The position of these points will, as we have before shown, coincide with the centres of gravity of the shaded portions; hence, by taking the moment of each portion round the neutral axis and dividing by the previously ascertained area $=2.49$ square inches, we obtain the required position of the centres of pressure, in the present instance 1.84 in. above, and 1.66 in. below the neutral axis. The effective depth of the rail will, consequently, be $1.84+1.66=3.5$ in.; and, since the effective area is 2.49 square inches, the moment of resistance of the cross section will be $M=3.5\times2.49=8.715$.

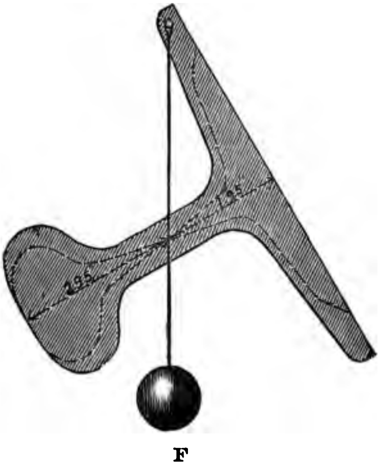
In the earlier part of this investigation we stated that the moment of resistance of any cross section, although apparently involving a tedious, if not difficult, process of computation, might, in reality, be determined with little, if any, calculation. We will now justify our assertion, and we think that, even if our aim of exhibiting more clearly to students the *rationale* of the several steps in the process of determining the moment of resistance of a given cross section, be not attained, the attempt will at least afford an amusing illustration of the saving in "skilled" labour which may be effected by simple "calculating machines."

Let a full-sized template of the rail cut out of cardboard, or, still better, out of tin-plate, be allowed to hang freely from a pin passing through a hole in the flange (Fig. 9), and from the same pin let a fine plumb line be suspended, then the intersection of the plumb line with the centre line of the rail previously drawn on the template will give the position of the centre of gravity of the cross section of the rail, and it may be easily demonstrated that the neutral axis will coincide with it in position.

Having thus ascertained experimentally the position of the neutral axis, the effective area of metal above and below that point may be exhibited on the template by geometrical construction, as already explained; and it is only necessary, then, to ascertain the area in square inches, and the centres of pressure of those figures, above and below the neutral axis, which may be done by the following mechanical process:

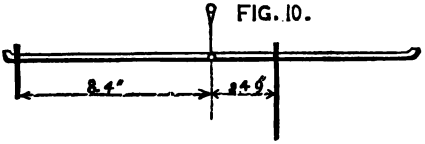
Let a straight piece of wire, the length of which in inches is equal to double the area of the rail in square inches, be suspended by a thread attached to its centre. (Since the sectional area of a rail in square inches is on the average, equal to $\frac{1}{10}$th the weight of the rail in pounds per yard, the length of the wire in the present case will be $\frac{2 \times 84}{10} = 16.8$ in.) To one end of this wire let the template of the rail be attached, and let it be balanced by a piece of card-board or tin-plate, as the

FIG. 9.



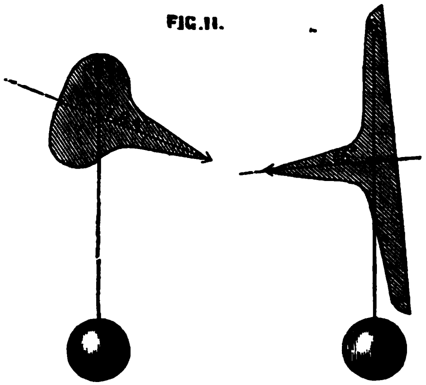
case may be, slipped on to the opposite arm of the lever. Then, if the template be removed, the figures of pressure previously traced on it cut out, and one of them attached to the vacant end of the wire, we can, by slipping the balance-weight along the wire till equilibrium is attained, ascertain the relative weights, and, therefore, areas, of the original template, and of the figure of pressure. For, since the length of one arm of the lever in inches was made equal to the area of the rail in square inches, and since the weights, or areas, vary inversely as the lengths of lever, the length of the other arm in inches from the point of suspension to the balance-weight will be equal to the area of the figure of pressure, that is, to the effective area of metal above and below the neutral axis in square inches. In this instance (Fig. 10) it will be found to measure 2.49 in.

FIG. 10.



It only remains now to ascertain the centres of pressure of the forces acting above and below the neutral axis, and the position of those points will be identical with the centres of gravity of the figures cut out of the template, which may be found by the process adopted when the centre of gravity of the entire section had to be determined (Fig. 11).

FIG. 11.



We have now arrived at the effective area=2.49 in., and the effective depth=3.5 in., without the use of a single figure; and the only computation necessary to obtain the moment of resistance of the rail will be the simple multiplication of one of those ascertained amounts by the other; so we think our proposition, that little, if any, calculation was absolutely indispensable, has been clearly demonstrated.

Having determined the moment of resistance of the cross section by one of the preceding methods, the transverse strength of the rail may be readily obtained. It must be premised, however, that since the maximum unit strain on the rail will, when the rail is in its normal position, that is, with the flange downwards, be compressive, whilst we need concern ourselves with tensile strains only, which will be less than the maximum compressive strains in direct proportion to the distance of the extreme fibres from the neutral axis, that is, as

1.95 : 2.55,—the equivalent value of the moment of cross section, as far as the tensile strains are concerned, will be:

$$M=\frac{8.715\times 2.55}{1.95}=11.4.$$

Now let us ascertain what weight the rail should support, applied at the centre of 60 in. bearings.

We have the moment of cross section, $M=11.4$; the ultimate apparent strain on the extreme fibres $F=38.3$ tons per square inch; and the moment of weight

$$(W)=m=\frac{W\times 60\text{ in.}}{4};$$

hence, since $m=M\,F$, we have

$$W=\frac{4\times 11.4\times 38.3\text{ tons}}{60\text{ in.}}=29.11\text{ tons}.$$

Five pieces of rail were tested at the bearing assumed above, with the following results:

Elastic Limit.		Ultimate.	
Weight.	**Deflection.**	**Weight.**	**Deflection.**
Tons.	Inches.	Tons.	Inches.
16	.45	30	9.4
"	.47	29.6	8.7
"	.54	29	10.5
"	.46	28.5	9.2
15¼	.49	28.75	14.8
Mean Result...	... ...	29.17	10.5

It appears, then, that the strongest rail tested gave a result 3 per cent. only in excess of the resistance previously deduced, whilst the weakest rail was some 2 per cent. below it. The average result of the tests (29.17

tons) may be considered identical with the calculated amount (29.11 tons).

An opportunity is afforded of putting the method of calculation advanced in this investigation to a crucial test, by comparing the calculated and actual breaking weight of the same rail tested head downwards. Under this condition the maximum strain on the fibres will be tensile, hence the effective moment of section will be $M = 8.715$, and the weight

$$W = \frac{4 \times 8.715 \times 38.3 \text{ tons}}{60 \text{ in.}} = 22.2 \text{ tons.}$$

Now, the results of actual test at Kirkaldy's were as follows:

Elastic Limit.		Ultimate.	
Weight.	Deflection.	Weight.	Deflection.
Tons. 15	Inches. .3	Tons. 22	Inches. 5.42

The difference between the experimental and calculated result is therefore less than 1 per cent. Hence, since no complicating conditions due to differences in the quality of the steel are involved, we may consider the reliable nature of the results given by the process of computation adopted to be satisfactorily illustrated.

It may be contended, with reason, that the strength of the rail deduced in the preceding investigation is of little practical value, since the rails prepared for fixing have a series of holes $1\frac{1}{16}$ in. diameter in the flanges, a condition which must obviously materially modify their ultimate transverse resistance. We shall find a much greater loss of strength is sustained than that due to the diminished sectional area alone; and if we measure results by the work done in effecting rupture, the detraction from the resistance induced by the holes will be far greater than would at first sight be considered possible. It will be well, therefore, to investigate the question of the probable strength of the rails with holes, in the same detailed manner already adopted in the instance of the solid rails; since, by so doing, it is not unlikely some desirable modification in the system of fastening may suggest itself.

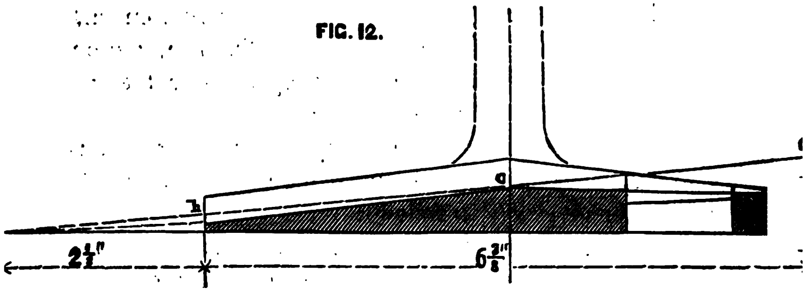
The first thing to be noted is the position of the holes, which are placed, not opposite each other, but 4 in. apart on alternate sides of the web of the rail; hence the cross section of the latter through the holes is unsymmetrical with respect to the centre line of the web.

Let us consider the effect of this condition. We have, in the bottom flange of the rail, a wide strap of metal with a tensile strain—transmitted through the web—applied in the direction of its centre line. Now, if the sectional area of metal on each side of the line of action of the pulling stress were equal in amount and symmetrically disposed, the unit strain on the metal would be uniform over the entire cross section; but if this condition be not obtained—as is the case with the rails we are considering—in order to establish equilibrium, the deficiency of sectional area on one side of the centre line must be compensated for by an increased unit strain on the metal on that side.

This proposition will be more clearly understood by its application in the present instance.

Referring to Fig. 12, representing a section of the

FIG. 12.



bottom flange of the rail, through a hole, subjected to a tensile strain, the centre of action of which lies in the plane of the centre line of the web; it will be apparent that, since the forces on each side of the centre line must balance, the deficiency of metal on the side of the hole will induce an increased strain per square inch on the edge of the flange nearest the hole, compared with that occurring on other portions of the same member. The *average* unit strain will, of course, be that due to the sectional area of the flange, irrespective of the *position* of the whole; consequently, the excess of strain on the metal on one side of the centre line will be accompanied by an equivalent reduction in the amount on the opposing side. Now, higher tensile strain per square inch involves greater elongation; consequently, under a direct pull, the flange of the rail would be deflected laterally towards the side on which the hole is placed, precisely the same as it would deflect if, in addition to the direct pulling stress, a bending force were applied opposite the hole, and acting in the direction of the arrow shown on Fig. 12. The mean unit strain on the fibres will be the same as that occurring on the fibres at the centre line of the rail, whilst on one side of that line the unit strain will diminish, and on the other side increase, in direct proportion to the distance of the fibre from the centre line.

The relative values of the mean, maximum, and minimum unit strains on the flange of the rail may be determined geometrically as follows:

Referring to Fig. 12: produce the base line of the flange to an assumed point, A, and from the opposite edge of the flange erect the perpendicular, B C, setting off on that perpendicular the heights, B E and B C, equal to the thinnest and thickest portions respectively of the metal in the flange. Then from A to C draw a line intersecting the centre line of the rail at the point *a*, and from A to E draw another line intersecting the edge of the flange at the point *b*. Joining the points *b*, *a*, E, we obtain the equivalent area of flange (shaded on diagram), assuming the unit strain to be uniform throughout, and equal in amount to the maximum stress occurring on any fibre. Now, since the centre of action of the pulling stress lies in the centre line of the rail, it is obvious the centre of pressure of the forces in action, or what we have shown in previous examples to amount to the same thing—the centre of gravity of the shaded portions on the diagram—must be in the centre line of the rail also. If upon trial this condition be not fulfilled, it will be necessary to assume a fresh position for the point A, and proceed as before.

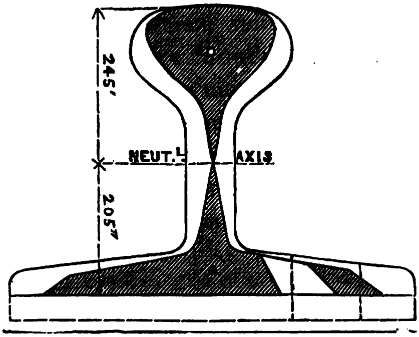
In the present instance, the position of the point A will be found to be ⅓rd the width of the flange from its edge: hence, since the strain on any fibre will be directly proportional to its distance from A, the maximum unit strain will take place on the edge of the flange near which the hole is made, and the minimum strain will, of course, occur on the opposite edge. In the former instance the unit strain will be $\frac{8}{5}$ths, and in the latter $\frac{2}{5}$ths, of the mean strain on the flange, as may be readily seen by scaling the distances of the centre line and of the two edges of the flange from the point A on Fig. 12.

We have assumed in the preceding investigation the flange of the rail to be a flat bar under a direct tensile strain, whereas, in point of fact, the tension being excited by a transverse stress will diminish in amount towards the neutral axis of the rail. A little consideration, however, will make it evident that the results arrived at in the former instance will hold good in the latter also, since the flange may be conceived to be made up of a number of layers, under varying degrees of tension, each of which will be affected in a similar manner to the flat bar assumed in the first instance; hence the maximum strain on the flange will also be $\frac{8}{5}$ths of the mean amount.

In order to determine the transverse strength of the rail with holes in the flange it will only be necessary, therefore, to ascertain the moment of resistance of the cross section without reference to the position to the hole; since we have already ascertained, in the present instance, the mathematical value of the maximum unit strain will be $\frac{8}{5}$th of the mean amount due to the moment of resistance of the cross section.

Proceeding by the method already exhibited when dealing with the solid rail, we find, Fig. 13, the position of the neutral axis to be 2.45 in. from the head of the rail; the centre of pressure being 1.76 in. and 1.67 in. above and below that point. The effective area of metal shaded on the diagram being 2.4 square inches, and the effective depth 1.76 in. + 1.67 in. = 3.43 in., the moment of resistance of the cross section will be M = 2.4 in. × 3.43 in. = 8.23; and from that we might at once deduce the maximum compression on the rail; but, as we need only consider tensile strain, it will be necessary to ascertain the effective moment of resistance of the portion of the rail below the neutral axis.

FIG. 13



Now, the *mean* value of the varying maximum tensile strain occurring on the outer layer of fibres in the flange will be less than the maximum compressive strain in direct proportion to the extreme distance of the respective fibres from the neutral axis; hence the equivalent moment for the *mean* tensile strain will be

$$M = \frac{8.23 \text{ in.} \times 2.45 \text{ in.}}{4.5 \text{ in.} - 2.45 \text{ in.}} = 9.82.$$

But the maximum unit strain on each layer of fibres will, as we have already shown, be increased to $\frac{8}{5}$ths the mean amount on account of the unsymmetrical position of the holes. Hence, since the maximum strain is the real measure of the strength, the effective value of the moment will be reduced in that ratio; consequently,

$$M=\frac{9.82\times 5}{8}=6.14;$$

and with that value of M alone will it be necessary for us to deal.

Having thus obtained the effective moment of the cross section, the next step is to ascertain the mathematical value of the maximum strain on the fibres. A little consideration will make it evident that, if the value of the moment of cross section is affected by the position of the holes, that of the maximum unit strain will also be affected, but to a much greater degree. For, by reference to Fig. 12, it will be seen that the angular deflection of the flange laterally under a tensile strain will be similar to that of a solid beam, of which the point A is the neutral axis, and the semi-depth A B; hence the depth of the solid beam would be $\frac{8}{3}$rds the width of the flange of the rail.

Now, in a previous investigation it was stated that the value of φ was dependent upon the depth of metal in the beam conjointly with the deflection; it follows, therefore, that its value in the present instance will be $\frac{3}{8}$ths of that obtained in a solid bar, since the width of flange represents the depth of metal in the beam of which, as we have already observed, the depth is equal to double the distance of the point A from the outer edge of the flange. In other words, the apparent maximum tensile resistance of the flange under a *direct* pull would be greater than the actual strain on the fibres, in consequence of the assistance the less strained fibres afford to the contiguous more severely strained ones, to the extent represented by the fraction $\frac{3}{8}\varphi$.

Again, the mean width of the flanges being $\frac{63}{16}$, and that of the web $\frac{11}{16}$, the value of φ will be the fraction $\frac{11}{63}$ of that obtained in a solid bar, which for the steel used in the present rails we have ascertained to be $.6f$. Hence the equivalent value of φ for the present section will be $\varphi=.6f\left(\frac{3}{8}+\frac{11}{63}\right)=.33f$. Taking $f=35$ tons per square inch as before, we obtain $\varphi=11.5$ tons, and the mathematical value of the maximum strain on the fibres $F=f+\varphi=46.5$ tons per square inch.

We have now all the necessary data for determining the transverse strength of the rails with holes as prepared for fixing. Taking the same bearing as before, that is, 60 in., the breaking weight applied at the centre would be

$$W=\frac{MF}{m};$$

by substitution,

$$W=\frac{4\times 6.14\times 46.5\text{ tons}}{60\text{ in.}}=19\text{ tons}.$$

Now, in this instance, as before, the results of direct experiment afford us an opportunity of testing the accuracy of the method of investigation, since two pieces of rails with drilled holes were tested at the bearing assumed in the previous calculation with the following results:

No. 1. Position of hole 7¾ in. from centre of bearing. Breaking load 27.6 tons.

No. 2. Position of hole 2 in. from centre of bearing. Breaking load 20.4 tons.

As in both instances fracture took place through the holes, although they were not made at the point of maximum stress—the centre of the bearing—as assumed in the previous calculation, it becomes necessary to reduce the actual breaking load to the equivalent amount had the holes occurred at the centre. Since the strain at any point of the rail would be directly proportional to its distance from the nearest bearing, the stress being applied at the centre, we have the following values for the required equivalent load:

$$\text{No. 1.}\quad W=\frac{27.6\text{ tons }(30\text{ in.}-7\tfrac{3}{4}\text{ in.})}{30\text{ in.}}=20.4\text{ tons.}$$

$$\text{No. 2.}\quad W=\frac{20.4\text{ tons }(30\text{ in.}-2\text{ in.})}{30\text{ in.}}=19\text{ tons.}$$

Mean result $=19.7$ tons.

It will be remembered the result arrived at by investigation was 19 tons, which is precisely identical with that obtained in the second experiment, whilst the mean variation is only some 3 or 4 per cent. In this instance, therefore, as in that of the solid rail, the results of investigation and direct experiment are practically identical.

It is hardly possible to inspect the diagram of the effective area of metal exhibited on Fig. 12, without the idea suggesting itself that the loss of section is greater than that which would have resulted from the making of an additional hole, equal in diameter and opposite to the single one at present made in the flange. By actual measurement, it will be found that without holes the effective area of flange is 2.82 square inches, and with *one* hole, 1.76 square inches; whilst, had two equal and opposite holes been made in the flange, the effective area would have been increased to 2.22 square inches; so that, other conditions being similar, the removal of a considerable quantity of metal from the flange would have had the paradoxical effect of increasing its power of resistance in the ratio of 2.22 to 1.76.

Further consideration, however, will show that this deduction, although generally true, is subject to correction. Thus, with *two* holes the *mean* unit tensile strain will be greater, because the neutral axis will approach nearer to the bottom flange; and as the unit strain on each layer of fibres will in this instance be uniform, the value of φ will be smaller than before; that is, the mathematical value of the ultimate resistance per square inch will be lower than it was in the case of the flange with one hole only. It obviously, then, depends entirely upon the size of the holes relative to the width of the flange whether a rail with holes opposite, or breaking centres, would offer the higher resistance. As it is desirable to know the practical bearings of the case, we will now investigate the question.

The moment of resistance of the cross section with two holes will be found by the method already exhibited in the two previous instances to be $M = 8.4$; and the value of $\varphi = .6 f \frac{11}{84}$; hence, when $f = 35$ tons per square inch, we have $F = 39.2$ tons per square inch. The breaking weight at the centre of 60 in. bearings will, therefore, be:

$$W = \frac{4 \times 8.4 \times 39.2 \text{ tons}}{60} = 21.9 \text{ tons.}$$

No experiments were made on the strength of rails with *drilled* holes placed opposite each other; but with *punched* holes the resistance was found to be $\frac{1}{12}$th higher with the holes so placed. Assuming, as we safely may, *the same to hold* good with drilled holes, the resistance would have been $1\frac{1}{12} \times 19.7$ tons $= 21.4$ tons, which again is practically identical with the result derived from theoretical considerations.

The following Table exhibits a summary of the results of our investigation on the transverse strength of the 84 lb. steel rail at 60 in. bearing:

84 lb. Rail.	Ultimate Strength.	
	By Experiment.	By Investigation.
	Tons.	Tons.
Solid Rail	19.17	29.11
,, inverted	22	22.2
Drilled holes breaking centres as adopted	19.7	19
Drilled holes placed opposite each other	21.3	21.9
Mean of all tested	23.04	23.05

Since the conditions under which these rails were tested were so varied, affecting both the moment of cross section and the varying value of φ, we may justly consider the striking similarity of the results given by investigation and by experiment to afford conclusive evidence that the method of investigation advanced may be relied upon as giving perfectly trustworthy results.

In taking the same lot of rails tested under such varied conditions—the solid ones, both in their normal position and inverted, and the others with differently placed holes—we have tested much more severely the correctness of the various hypotheses made than we should have done by taking different lots of rails of varied sections. The remarkable uniformity in the results of the several sets of experiments proves, beyond suspicion, that the steel in the whole lot was of equally good quality, and it follows from this that the identity of the calculated and experimental results could not be fictitious.

We think no method of calculating the transverse strength of a rail reliable which does not take into consideration, in the first place, the direct tensile resistance of the steel proposed to be employed; in the next place, the value of φ, which will be governed partly by the hardness of the steel, and partly by the form of cross section; and, lastly, the correct mathematical value of the moment of resistance. To exhibit the error which would have resulted from the omission of one of these considerations, the variable value of φ, we have calculated the following Table, showing the breaking weights, on the hypothesis that the rails would in all cases fail, when the maximum tension on the fibres was of the intensity sufficient to induce fracture under a direct tensile stress:

84 lb. Rail.	Ultimate Strength.	
	By Experiment.	By Calculation.
	Tons.	Tons.
Solid Rail	29.17	26.6
" inverted	22	20.3
Drilled holes breaking centres as adopted	19.7	14.3
Drilled holes placed opposite each other	21.3	19.6
Mean of all tested	23.04	20.2

It appears, then, that in the system actually adopted —drilled holes breaking centres—the error in the calculated result on the preceding false hypothesis amounts to 30 per cent., and that the average error is not less than 12½ per cent., whilst it may be remembered, on the former hypothesis the results were identical with those deduced from experiment.

On testing a set of rails with *punched* holes, breaking centres as in the rails with *drilled* holes already illustrated, a striking anomaly was presented. In all experiments the ultimate transverse strength of the rail was far below the amount due to the sectional area and quality of the steel, which was of the same tough nature throughout. The rails were placed in the machine with the holes at varying distances from the centre of the bearings; consequently since fracture in all instances occurred through the hole nearest that point, in order to institute a comparison, it becomes necessary to reduce the observed breaking stresses to the equivalent amounts had the holes occurred at the point of maximum stress.

The following Table exhibits the observed, and equivalent reduced, ultimate resistance of each specimen experimented upon, the stress being applied as before at the centre of 60 in. bearings:

Observed.		Reduced.	
Stress.	Deflection.	Stress.	Deflection.
Tons.	In.	Tons.	In.
18.25	1.76	13.5	.35
17	.7	13.4	.35
14.4	.38	13.5	.37
14.2	.35	13.3	.34
14.2	.35	13.3	.34
15.75	.49	13.4	.35
Average result		13.4	.35

Since the uniformity of the reduced ultimate stresses clearly proves that the reduction of the resistance from 19.7 tons to 13.4 tons—a loss of 32 per cent. in the *punched* rails as compared with the *drilled* ones—could not have resulted from any exceptional defect in the quality of the steel, the conclusion is irresistible that the mechanical action of punching must have deteriorated the metal to a degree corresponding to that percentage. In fact, this deduced result is amply confirmed by numerous other experiments on the direct tensile resistances per square inch of solid and punched steel plates recorded by Mr. Reed in his work on shipbuilding. It was found, also, by Mr. Sharp, of the Bolton Iron and Steel Works, that the average ultimate resistance to tension of $\frac{3}{16}$th Bessemer steel plates was 36 tons per square inch when drilled, and 24 tons only when punched, thus exhibiting a loss of no less than 50 per cent. due to punching. Subsequent experiments on drilled and *annealed* punched plates $\frac{5}{16}$th thick, gave an average ultimate resistance of 41 tons per square inch in each instance. With mild steel plates the loss by punching, and consequently the benefit of annealing, is not so marked, thus in some $\frac{3}{8}$th crucible steel plates the gain per cent. of annealed over unannealed was found to be 14 lengthways and 12 crossways. A large proportion of the loss due to punching may be obviated if the die employed be made a certain amount larger than the punch; thus it was found by Mr. Sharp that with ordinary holes, some $\frac{1}{2}$ in. plates broke with 26 tons per square inch, whilst with the taper holes obtained by using a die $\frac{3}{16}$th larger in diameter than the punch, the ultimate resistance was 32½ tons per square inch, showing a difference of 25 per cent. in favour of taper punching.

It appears, then, that the tensional resistance of the steel employed in the manufacture of the Metropolitan rails after the process of punching, would be only 36 tons per square inch less 32 per cent.=24½ tons per square inch. Now iron in ordinary rails will sustain about 22 tons per square inch, hence the advantage of the superior strength of steel is almost neutralised when the holes in the flanges are punched in the ordinary manner, without subsequent annealing.

If the resistance of the rail to a *steady* transverse stress be diminished to the extent indicated by the foregoing experiments, it may easily be imagined that its power of resisting a blow will be far more seriously detracted from, and for the following reasons:

The steel in the immediate neighbourhood of the hole alone being injured, at that point it will be giving way, when at all other points of the beam it will be strained little if any beyond the elastic limit. It follows that the deflection, and the "fatigue," under repeated applications of stress, will be concentrated at the hole; and it also follows, from the fact that the deflection within the elastic limit is but a small fraction of the ultimate deflection, that the amount of the latter will be smaller in a punched rail than in a drilled one, and much smaller again in the drilled rail than in a solid one.

The specific amounts determined by experiment are as follows:

Ultimate Deflection at 60 in. bearing.

Solid Rail.	Drilled Rail.	Punched Rail.
10.5 in.	2.4 in.	.35 in.

Now the respective breaking weights being 29.17 tons, 19.7 tons, and 13.4 tons, the work done in effecting rupture in each case would be proportional to the product of the weight by one-half the corresponding deflection, consequently in foot-tons it would have the following values:

Solid Rail.	Drilled Rail.	Punched Rail.
12.76	1.97	.195

The drilled rails thus offering upwards of 10 times, and the solid ones 65 times, the resistance of the punched rails.

The ultimate deflection of the punched steel rail is precisely the same as that of a cast-iron one of similar section, the weight required to produce these deflections being in the ratio of about 2½ to 1. The amount of deflection—less than ⅜ths of an inch—is so small that imperfect packing of the sleepers might easily permit it to take place; so that if the weight on the driving wheels of a locomotive were some 60 per cent. greater than it is at present, fracture would occur sooner or later in almost every instance. As things are, a punched rail would be strained beyond the limit of elasticity by the passage of an ordinary engine, if an intermediate sleeper sank but ⅙th of an inch. It is palpable, then, if there be the remotest chance of a solid steel rail fracturing under exceptional conditions, the liability to accident in the instance of a punched rail will be of serious moment.

With reference to the rails for the Metropolitan Railway, the weight of these considerations was sufficient to induce Mr. Fowler to prohibit the use of punched rails; and there can be no question, that even if the difference of cost between punching and drilling were an important item, which it is not, this decision was wisely made.

It was premised before investigating the question of the transverse strength of rails with holes in the flanges, that some desirable modification in the system of fastenings might suggest itself, and such has proved to be the case. In the first place, it has been amply demonstrated that punching cold without final annealing is inadmissible at any price; and it is also evident that if the diminished amount of elasticity possessed by the drilled rails be sufficient to ensure safety under severe traffic, a very much harder and, consequently, more durable, rail might be used with equal safety if holes were dispensed with altogether, and another mode of fastening adopted. If hard wood sleepers could economically be adopted in the permanent way of our own railways, the flanges of steel rails might be much reduced in width, and, probably, coach screws and clips would form a most efficient and durable fastening. This, however, is one of the many questions which must be left to the judgment of the engineer, guided by the conditions of the specific case. No general rule could with propriety be laid down. Simplicity of parts is a great desideratum, and this is attained to the greatest practicable extent in Mr. Fowler's system. There is, moreover, no reason to doubt that ample security combined with a high amount of endurance has also been secured.

It has been urged with reason that the conditions affecting the construction and maintenance of permanent way are so modified by the immense increase in the durability of the rail itself, when made of steel, that, in all probability, the old double-headed type will soon become extinct. Undoubtedly, if a steel rail lasts sufficiently long to obviate the necessity of turning, the complexity of the parts in the latter section is not justified by any compensating advantage. As, however, double-headed steel rails are at present pretty extensively used, and are likely to be so for some time, a little space may, with propriety, be devoted to exhibit the application of the general principles and methods of calculation advanced in the preceding investigations to that form of rail. For this purpose we may take the rails of the Great Indian Peninsula Railway, manufactured by Sir John Brown and Co., who have kindly furnished us with particulars of the numerous tests carried out on those rails.

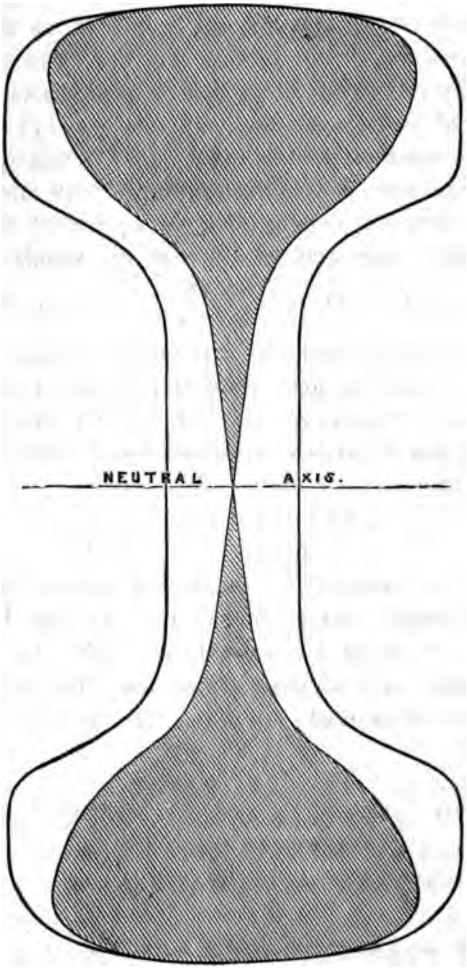
The section of rail, the weight of which is 68 lb. per yard, is shown full sized on Fig. 14. Calculating the moment of resistance of the cross section, we find the effective depth to be 3.6 in., and the effective area 2.17 square inches. We have, therefore, the moment of resistance, $M = 7.8$, and the ultimate stress (W) at the centre of the span

$$(S) = W = \frac{4 \times 7.8\,F}{S}.$$

Hence, to determine the transverse strength of the rail it is only necessary to know the value of $F = f + \varphi$.

The resistance to direct tension (f) was ascertained experimentally, the ultimate amounts per square inch in nine instances being as follows: 50, 37, 38, 45, 48, 47, 50, 44⅔, and 45½ tons, the average of the whole being 45 tons per square inch.

FIG. 14.



This high tensile resistance shows the steel in these rails to be much harder than that in the flanged rails of the Dowlais Company; and it follows, as we have already frequently pointed out, that the value of φ will necessarily be increased. It may be remembered in the instance of the Dowlais rails we took $\varphi = .6f$; and we may conclude that for these hard rails it will approach its ultimate value of $.8f$. Assuming its true value for a solid bar of this steel to be $.75f$, the equivalent value for the rail will, since the thickness of the web is $\frac{11}{16}$, and the width of head $\frac{40}{16}$, be $\frac{.75 \times 11}{40} f$; hence $\varphi = .206f$, or since the average value of f by experiment was 45 tons per square inch, we have $\varphi = 9.27$ tons, and $F = f + \varphi = 54.27$ tons. Substituting this value of F in the previous equation, the ultimate breaking weight at the centre of a span of 43.5 in. would be:

$$W = \frac{4 \times 7.8 \times 54.27 \text{ tons}}{43.5 \text{ in.}} = 38.9 \text{ tons.}$$

Now, the average of forty-five direct experiments on the transverse strength of the rail at these bearings gave 38.7 tons as the ultimate strength. In this instance, then, as in all the previous ones, the results derived from theoretical investigation and experiments are identical.

The first part of our subject is now brought to a conclusion. In order to arrive at some reliable method of computing the probable strength of cast-iron, wrought-iron, or steel beams, of any given form of cross section, we first derived certain *data* from the analysis of a large series of experiments on the transverse strengths of simple bars. The variance in the proportional strengths of different forms of bars suggested the existence of a certain relationship between the apparent tensile resistance of the extreme fibres in a beam and the form of cross section; a provisional hypothesis was, consequently, made, and the transverse strengths of more or less complicated sections of beams were computed. The results thus deduced were compared with those derived from direct experiments; and as in no instance was any essential difference observable, the practical accuracy of the hypothesis upon which the calculations were based was considered to be satisfactorily demonstrated.

It is, of course, impossible to predict the breaking weight of a beam unless we know something of the quality of the metal used in its construction. But, at the same time, it is equally impossible to foretell the ultimate tensile resistance of a simple bar, or even of another portion of a bar which has already been submitted to actual test. Mr. Fairbairn's experiments show that the strength of the same bar varies at different cross sections; and the experiments on Krupp's steel shaft illustrate the fact that the same variation is exhibited in the strength of different fibres occurring in any given plan of cross sections of a bar.

The aim of the preceding investigation has been to prove that, sufficient *data* being advanced to enable the quality of the metal to be determined, the strength of any form of beam might be computed to as great a degree of accuracy as that of any simple bar subject to a pulling stress only. The various experiments have been selected with the view of making it, as far as possible, self-evident that bending stresses and direct stresses are, in all instances, convertible terms.

The preference maintained by us for the former method of applying the strain, as a test of the quality of metal in any given specimen, is based upon the consideration that in the instance of a beam the rate of extension and depression of the material within and beyond the elastic limit is exhibited in a greatly exaggerated, and consequently more measurable, form than it possibly could be in any bar of reasonable length subjected to direct stresses only.

Such being the case, our next step must be to ascertain the relationship existing between the comparative degrees of elasticity developed under these different conditions; so that we may be enabled to deduce one from the other, in the same manner as we have proved the respective ultimate resistances may be deduced.

PART II.

ON THE STIFFNESS OF STEEL RAILS AND OTHER ANALOGOUS BEAMS.

Section 7.

ON THE STIFFNESS OF CAST IRON.

A KIND of fatality seems to attach to all simple and fascinating generalisations, since, like Kepler's laws, they almost always turn out, upon more careful investigation, to be only broadly true.

Till a recent period any schoolboy would have been considered competent to estimate the strength of any given section of beam; for, knowing the resistance of a bar 1 in. square, was not that of a 2 in. square bar 8 times, and of a 3 in. bar 27 times, that amount; and was not the strength of a hollow shaft, or of a girder section, equal to that of the solid beam, minus the proper resistance of the portions removed?

We have seen to what fallacious conclusions these simple generalisations lead; how that, for cast or wrought iron, the specific size of the casting or forging modifies the *direct* unit resistance of the metal; and how for all materials the *form* of the cross section influences the *apparent* resistance of the same.

If this be true with respect to the *strength* of a beam, we may reasonably anticipate that still more perplexing conditions will affect any attempted generalisation of the varying *deflections* of a beam under different loads. On the ordinary hypothesis problems concerning deflection are readily solved, since it is a consequence of the assumed law of elasticity—*ut tensio sic vis*—that the deflection of the same beam will be directly proportional to the transverse stress. This assumption, however, is not strictly true for any material, and in cast iron the deviation from the truth was found by Mr. Hodgkinson to be considerable under every degree of strain.

The extension and depression of cast iron, under gradually increasing pulling and thrusting stresses, was determined by that able investigator with a considerable degree of precision. The results of his experiments on 1 in. square bars, 50 ft. long for the tensile and 10 ft. long for the compressive strains, are tabulated below; the loads per square inch are given in tons, and the corresponding increments and decrements in term of the length:

Tons per sq. in.	Extension.	Depression.
½	.000081	...
¾	.000123	...
1	.000167	.000166
1½	.000258	...
2	.000352	.000348
2½	.000450	...
3	.000552	.000539
3½	.000661	...
4	.000774	.000720
4½	.000895	...
5	.001022	.000895
5½	.001155	...

Tons per sq. in.	Extension.	Depression.
6	.001303	.001083
$6\frac{1}{2}$	.001501	...
7	.001674	.001274
8	...	.001471
9	...	.001665
10	...	.001854
12	...	.002246
14	...	.002673
16	...	.003180
18	...	.003703

It appears, then, that the ultimate extension of cast iron averages $\frac{1}{600}$th of the length, and that the depression under the same strain averages $\frac{1}{775}$th of the length. The permanent set was ascertained to increase from $\frac{1}{30}$th the extension at 1 ton per square inch to $\frac{1}{7}$th the ultimate extension: and for compressive strains of similar amount from $\frac{1}{40}$th to $\frac{1}{10}$th the depression.

If the extension and compression of fibres under strains excited by transverse stresses were the same as it is under direct strains, with the aid of the preceding Table the deflection of a cast-iron bar under a given transverse stress might easily be computed. We have found, however, many anomalies in the theory of transverse strength, and we may expect the same to obtain in that of deflection.

The first question suggesting itself is this: Will the deflection be that due to the *actual* or to the *apparent* strain on the fibres, or will it be governed by some other condition conjointly with one of those? This question could only be settled by a series of experiments arranged with that view, but those carried out by Mr. Barlow, for another purpose, throw a little light upon the subject. A beam 7 ft. 4 in. long, 6 in. deep, and 2 in. thick, was subjected to a stress of 5786 lb. applied at its centre, when the extension of the outer fibres, measured by a micrometer, was ascertained to be $\frac{1}{1792}$nd of the length. Now the *apparent* strain on the fibres under that load would be $F=(f+\varphi)=10{,}608$ lb. per square inch, and a *direct* strain of that amount, it will be seen by referring to the preceding Table, extends cast iron $\frac{1}{1056}$th of the length, or nearly double that exhibited by the beam. In another beam a stress of 8000 lb. produced an extension of $\frac{1}{1207}$th of the length; the strain on the outer fibres under that stress being $F=14{,}666$ lb., and the extension due to the direct application of a strain of that intensity per square inch being $\frac{1}{848}$th of the length, or, again, nearly double that in the beam.

Now, in our analysis of the strains on a rectangular cast-iron bar, subjected to transverse stress, we found the *actual* strain on the fibres to be less than half the apparent strain; and we now find the *extension* of the fibres to be very nearly proportional to the actual strain. It would, apparently, follow from this ascertained condition that the deflection of a beam would also be proportional to the maximum *actual* strain on the fibres; we shall find, however, this conclusion reversed by the results of direct experiment, the actual deflection of a beam being very nearly proportional to the maximum *apparent* strain on the fibres. That is to say, although it has been proved by actual measurement that the ultimate extension of cast iron before rupture is about the same in a beam as in a simple bar under a direct pull, yet the effective elongation of the fibres in a beam, as far as deflection is concerned, may be considered greater than the actual amount in the same ratio that the *apparent* strain exceeds the *actual* strain.

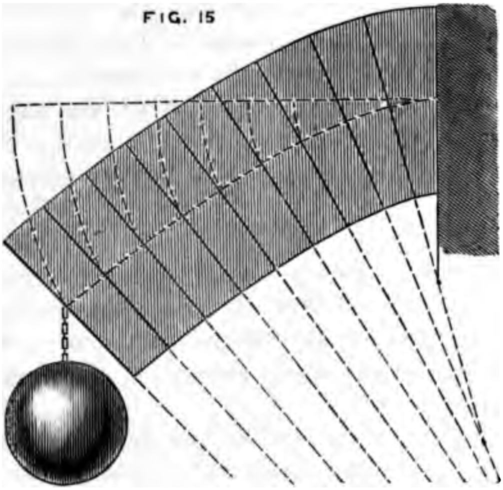
We do not propose to theorise on this subject, or to attempt any explanation of the anomaly. Having pointed out the existence of this complicating condition, we shall content ourselves with deducing practical results from direct experiments.

In order to institute a comparison between the extension and compression of cast iron under direct strains, and under strains excited by transverse stresses, we must briefly consider the *modus operandi* of the elements in the latter instance.

It must first be premised that the deflection of a beam subjected to a transverse stress applied at its centre will, under a given strain per square inch, be identical with that of an overhanging beam half the length $\left(\frac{S}{2}\right)$, fixed rigidly at one end.

Now let us conceive such a beam (Fig. 15) divided into an infinite number of lengths; then upon the application of a weight at the end of the beam, the top edge would be extended, and the bottom compressed, so that the segments would be transformed into a series of wedges, the taper of each being directly proportional to its distance from the point of support. It will be apparent from an inspection of the diagram that, since each wedge tilts down the entire portion of the beam extending to the free end, the fraction of the total deflection due to each wedge will be inversely proportional to its taper, and directly proportional to its distance from the end; that is to say, the deflection due to the action

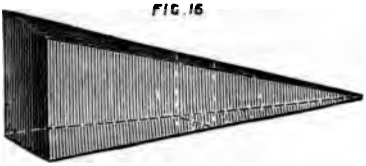
FIG. 15



of each wedge will be proportional to the square of its distance from the free end of the beam.

Representing the amounts of deflection by the contents of a solid, the resulting figure (Fig. 16) will be

FIG. 16



a pyramid, the length of which is equal to that of the beam, and the area of which at any point will be proportional to the amount of deflection due to the segment at that point. Consequently, if the area of the base represent the deflection assuming the unit strain at that *point to be ob*tained throughout the entire beam, and assuming also the whole of the extension and compression to be concentrated there, then the mean area of the pyramid—which may easily be shown to be equal to $\frac{1}{3}$rd that of the base—will represent the actual deflection of the beam.

Now S being the span of a beam supported at each end, $\frac{S}{2}$ will be that of the equivalent overhanging beam, with which, for the sake of simplicity, we have been dealing. Again, e being the mean value of the ultimate extension + ultimate depression per unit of length, $S\,e$ will represent the total amount on the assumption that it will be uniform throughout the entire beam; and assuming it to be concentrated at the fixed end of the overhanging beam, the deflection will be greater in the inverse ratio of the arms of the bent lever constituted by the depth (d) and the span $\left(\frac{S}{2}\right)$. The deflection on the preceding assumptions would, therefore, be $\delta = \frac{S^2 e}{2d}$; but we have already shown that the actual deflection will be less in the same proportion that the mean area of a pyramid is smaller than that of the base; hence, the true ultimate deflection of a beam supported at each end and loaded at the centre will be: $\delta = \frac{S^2 e}{6d}$.

The value of e from direct experiment may be abstracted from the Table of extensions and depressions already given. Thus, the extension per unit of length at the average ultimate tensile resistance of 7 tons per square inch equals .00167; and the depression under the same force equals .00127; hence,

$$e=\frac{.00167+.00127}{2}$$

$=.00147$. We will now deduce its value from experiments on transverse strain, and, for the purpose, will take those made by Mr. Hodgkinson for the Iron Commissioners:

Span, S.	Depth, d.	Ultimate Deflection, δ.	Ultimate Mean apparent Extension and Depression per Unit of Length $=e=\frac{6d\delta}{S^2}$.
In.	In.	In.	
162	3	4.627	.0032
162	6	2.4	.0033
162	1½	9.	.0031
81	3	1.29	.0035
81	1½	2.44	.0033
108	1	5.	.0026
108	2	2.85	.0030
54	2	.84	.0034
54	1	1.78	.0037

Mean result ... ... $e=.00323$

It appears, then, that the mean *apparent* ultimate extension and depression of the fibres in a rectangular cast-iron beam is the fraction .00323 of the length; whilst by direct experiment the *actual* amount has been ascertained to be .00147 of the same. Hence the actual deflection of the beam is about $\frac{.00323}{.00147}=2\frac{1}{5}$ the amount derived from theoretical considerations alone. Now, we have already shown that the ultimate *apparent* strain on the fibres averages $2\frac{1}{4}$ times the actual resistance to direct tension; hence, it follows that the deflection of a cast-iron beam will be the same as it would be, were the apparent strain on the fibres actually obtained.

In deducing a formula for the deflection of a cast-iron beam, it is hardly necessary to complicate the question by the introduction of the varying rates of deflection under successive increments of load; since the result will be a sufficient approximation to the truth for all practical purposes, if we take the mean rate. Referring to the Table of elongations and depressions, the alterations of length due to successive additions of load will be as follows:

	Extension.	Depression.
1st ton	.000167	.000166
2nd ton	.000185	.000182
3rd ton	.000200	.000184

Mean result per ton=.00018 per unit of length.

To obtain the deflection of a solid cast-iron beam loaded at the centre to any extent within the elastic limit we have, therefore, merely to insert the preceding value of E in the equation

$$\delta = \frac{S^2 F E}{6d};$$

and the *ultimate* deflection of the beam will be given by the formula

$$\delta = \frac{S^2 e}{6d}, \text{ when } e = .00323.$$

To render the latter formula applicable to beams of various forms of cross section some little modification is necessary. Thus the value of F in a rectangular cross section, at the point of fracture, has been found to be 2¼ times the ultimate direct tensile resistance of the metal, which we have taken at 7 tons per square inch. Hence, since $F\,E = e$, we have

$$E = \frac{.00323}{2\tfrac{1}{4} \times 7} = .00021.$$

Substituting this value in the latter equation, and reducing, we obtain the following formulæ for the deflections of cast-iron beams of any symmetrical form of cross section:

$$\text{Deflection within elastic limit} \quad \delta = .00003 \frac{S^2\,F}{d}.$$

$$\text{Ultimate deflection} \quad \ldots \quad \delta = .000035 \frac{S^2\,F}{d}.$$

We may check the results given by the first of these formulæ by applying it in the instance of some thin bars tested by Mr. Tredgold. The size of the bars was 1.3 in. wide by .65 in. deep, and they were tested by a weight of 162 lb. applied at the centre of 35 in. bearings, the resulting deflection averaged .27 in.

$$\text{We have therefore, } \delta = .00003 \frac{35^2\,F}{.65},$$

$$\text{when } F = \frac{6 \times 35 \times 162 \text{ lb.}}{4 \times .65^2 \times 1.3 \times 2240}.$$

Hence, $\delta = .37$ in. by the formula; whereas the observed deflection was $\frac{1}{10}$th of an inch less.

We anticipated this result, and selected these experiments of Tredgold in order to exhibit it. The constant in our formula was deduced from bars varying from 1 in. to 3 in. in thickness, and we need not be surprised to find it give too high a result when applied to thinner and consequently very much harder bars. The falling off in elasticity, inseparable from greater thickness of metal, was well illustrated in the experiments from which our constant was derived. Thus the ultimate deflection of similar beams would, if the degree of elasticity were constant, vary as their linear dimensions; hence the deflection of the 54 in. × 1 in. bar being 1.78 in., that of the 108 in. × 2 in. should have been $1.78 \times 2 = 3.56$ in., and that of the 162 in. × 3 in. bar $1.78 \times 3 = 5.34$ in., whereas the actual deflections of the latter bars were 3 in. and 4.62 in. respectively. The ultimate strengths also should have varied as the squares of the linear dimensions, that is, the respective breaking weights should have been 440 lb., 1760 lb., and 3960 lb., whereas the observed amounts were 440 lb., 1338 lb., and 2861 lb., thus exhibiting a falling off in strength as well as in elasticity.

We may now pass on to the consideration of the elasticity of wrought iron and steel.

Section 8.

ON THE STIFFNESS OF WROUGHT IRON.

Numerous and careful experiments have been carried out with the view of ascertaining the rate of extension of wrought iron under tensile strains. Long bars are essential to a correct conclusion: even a length of 10 ft. is rather small; but experiments on bars much shorter than that should be carefully excluded from any complete generalisation. Mr. Hodgkinson tested two bars, 50 ft. long, the one being about ½ in. and the other about ¾ in. diameter, and found the extension per ton strain per square inch, in terms of the length, to average .0000804 for the first, and .0000809 for the second bar, the rate of extension being tolerably uniform up to half the breaking weight of the specimen.

Mr. Edwin Clarke gives the following results of an experiment on a bar 10 ft. long and 1 in. square:

Strain Ton. per sq. in.	Extension in terms of the length.
1	.000069
2	156
3	238
4	319
5	399
6	480
7	560
8	640
9	720
10	800
11	890
12	.001020
13	1280
14	2310
15	4160
16	4430
17	.010150
18	.012120
19	.020170
20	.021460
21	.024720
22	.034250

Professor Barlow tested a number of bars about 100 in. long, and of various sectional areas. The reduced results are exhibited in the following Table:

Gross Strain per sq. in.	Extensions per Ton Strain per Square Inch in Terms of the Length.						
Tons.	Bar 1 in. Square.	1 inch Square.	1 inch Diameter.	1 inch Diameter.	2 inch Square.	2 inch Square.	2 inch Square.
2	...	...	.0001250	.0001170	.0001800	.0001500	.0001250
3	.0000625	...	1170	1020	1100	1000	500
4	935	.0000800	1020	1090	1100	850	500
5	1090	900	940	1090	1000	750	1000
6	1100	1100	860	1020	1000	750	950
7	982	900	940	780	950	800	950
8	935	800	940	640	950	950	900
9	982	1000	940	...	750	...	900
10	947	800	940	...	1450	...	950
	.0000982	.0000903	.0001010	.0001010	.0001082	.0000957	.0000841

Average extension per ton = .0000964.

Very many other experiments have been made, but it is unnecessary to refer to them here. By general consent, the extension of wrought iron within the elastic limit, which ranges from some 8 to 12 tons per square inch, is taken at .0001 per ton; that is $\frac{1}{10000}$ of the length for each ton per square inch direct tensile strain.

It is now necessary to deduce the value of the same element from experiments on transverse stress so that we may be enabled to ascertain the laws of deflection affecting wrought-iron beams.

It has been shown that the deflection of a beam of uniform section, under a stress applied at the centre of the bearings, is

$$\delta=\frac{S^2 FE}{6d};$$

hence, since

$$F=\frac{S w}{4 M},$$

we have

$$E=\frac{24 M d \delta}{S^3 w},$$

which, for a rectangular cross section, becomes

$$E=\frac{4 d^3 b \delta}{S^3 w}.$$

We can, therefore, at once deduce the value of E—that is, the effective rate of extension per ton per square inch *apparent* strain from the following experiments on the transverse strength of various-sized bars:

Span. (S)	Depth. (d)	Breadth. (b)	Weight. (w)	Deflection. (δ)	Extension per Ton per Square Inch. (E)
In.	In.	In.	Tons.	In.	
25	2	2	3	.079	.000109
33	”	”	1	.04	.000072
”	”	”	”	.044	79
”	”	”	”	.05	89
”	”	”	”	.051	91
”	”	”	”	.060	.000107
”	”	”	”	.046	.000082
”	”	”	”	.064	.000114
”	”	”	”	.055	.000098
”	”	”	”	.048	86
”	”	”	”	.049	88
”	3	1½	”	.022	.000100
”	”	”	”	.026	119
”	2½	”	”	.04	100

Mean value of E=.000095

It is proved by the preceding experiments that the effective extension of wrought iron per ton apparent strain is the same in a solid beam as it is in a simple bar; it only remains, therefore, to ascertain whether this conclusion will hold good with respect to other sections of beams. The following results deduced from Professor Barlow's experiments on iron rails show that the same law does obtain in all sections.

Double-headed 60 lb. rail, 4½ in. deep. Moment of resistance of cross section M=6.7. Tested at 33 in. bearings. Three experiments:

Gross Load.	Deflection for each Ton.			
Tons.				$E=\frac{24 M d \delta}{S^3 w}$. Hence, E=.0001.
2	.004	.005	.006	
3	.005	.005	.004	
4	.003	.005	.005	
5	.005	.005	.004	
6	.004	.003	.003	
7	.004	.004	.004	
8	.005	.005	.004	
9	.006	.006	.005	
10	.007	.005	.006	
11	.007	.007	.008	
Average per ton	.005	.005	.0049	

Double-headed 75 lb. rail, 5 in. deep, tested at 4 ft. 6 in. bearings:

Gross Load.	Deflection for each Ton.		M=8.05.
Tons.			$E=\frac{24 M d \delta}{S^3 w}$. Hence, E=.000102.
3	.017	.013	
4	.008	.017	
5	.017	.023	
6	.015	.017	
7	.015	.019	
8	.020	.015	
9	.023	.017	
Mean deflection per ton ...	...	.0165	

No further evidence is required to prove that the value of E is constant for all cross sections, and that the extension of fibres under similar *apparent* unit strains is unaffected by the agency exciting those strains. We have, therefore, for the deflection of any wrought-iron beam of uniform section, within the elastic limit, under a stress applied at the centre,

$$\delta=\frac{S^2 FE}{6d};$$

hence, since E=.0001, we have:

$$\delta=.0000166\frac{S^2 F}{d}.$$

The determination of the *ultimate* deflection of a wrought-iron beam is a problem, the conditions of which are so complicated that no definite solution could possibly be given. The rate of elongation of the metal under strains approaching the ultimate limit is so irregular, even in the same bar, that no other conclusion could with propriety be arrived at. It may be advantageous, however, to consider briefly some of the conditions of the problem.

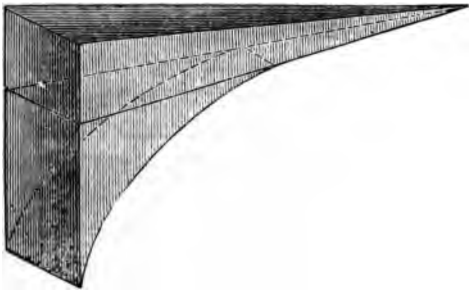
The ultimate deflection of a wrought-iron beam will obviously be governed to a great extent by the ultimate elongation of the material composing it; and we have already pointed out that the extension of various makes of iron at the time of fracture varies from some 5 to 30 per cent. of the length. That the same per-centage of elongation does not obtain in bars of varying lengths, other conditions being the same, was evidenced in the experiments of Mr. Fairbairn on some rivet iron, where it was found to vary from 21.6 per cent. in 10 ft. bars to 42 per cent. in 10 in. ones, and where, generally, the ultimate elongation (l) per unit of length (L) was found to be represented by the expression $l=.18+\frac{25}{L}$.

Kirkaldy has shown this apparent anomaly to be due to the drawing out of the metal at the point of fracture. Thus the ultimate elongation of a piece of Govan bar, tested by him, was 20 per cent. of the length, of which amount upwards of 6 per cent. occurred at the point of fracture; the remainder, some 14 per cent., it is urged, would be constant for all lengths of bars if of uniform quality. Another specimen of Bowling iron, with the same ultimate elongation, drew out nearly 8 per cent., whilst a bar of Bradley iron, with 29 per cent. ultimate elongation, drew out 13 per cent. at the point of fracture. Other bars, in which the drawing out was insignificant, gave tolerably uniform rates of elongation for all lengths.

These experiments are of very great weight in any attempted approximation to the ultimate deflection of a bar of wrought iron; because, if the iron be of a nature to draw out at the point of fracture, that is, at the centre of the bar—the very position, as we have seen, where it is most influential in producing deflection—the amount of the ultimate deflection of the bar will be very much greater than that due to the mean ultimate elongation of the iron. This condition explains why small bars and plates of tough iron may be readily folded close without fracture.

The preceding considerations indicate that the ultimate deflection of a wrought-iron beam will be governed by three conditions,—namely, the elastic extension, the mean ultimate extension, exclusive of the drawing out, and the latter element itself. In average iron the rate of elongation is practically uniform up to a strain amounting to about half the ultimate resistance; beyond that limit the rate rapidly increases, exclusive of drawing out, very nearly as the square of the additional strain. Representing the resulting deflection by the contents of a solid as before, we have (Fig. 17) a pyramid representing the deflection if elasticity perfect, and another solid, representing the actual excess of deflection due to deficiency of elasticity, exclusive of that due to drawing out. It is obvious, then, if the base of the pyramid represents the total elongation on the first assumption, that of the other solid will be proportional to the excess; that is, putting e for the mean ultimate elongation, the respective areas will be as F E and $e-$F E. But, from a number of experiments carried out by Kirkaldy on the rate of elongation under high strains, the mean area of the solids in each instance appears to be equal to $\frac{1}{3}$rd that of their bases; hence, since the length of one is double that of the other, the relative deflections due to each will be as $\frac{1}{3}$rd area and $\frac{1}{6}$th area. Taking the rate of depression as equal to that of extension, we have for the ultimate deflection, exclusive of drawing out,

FIG. 17.



$$\delta=\frac{S^2\,FE}{6d}+\frac{S^2\,(e-FE)}{12d};$$

hence

$$\delta=\frac{S^2\,(e+FE)}{12d}.$$

Applying this formula in the instance of some 2 in. square bars, tested at 25 in. bearings, we have $S=25$ in., $F=30$ tons per square inch, $E=.0001$, and, since the elongation of the same iron under a direct tensile strain, exclusive of drawing out, was 3 in. in a length of 15 in., we have also $e=.2$; hence,

$$\delta=\frac{25^2\,(.2+.003)}{12\times 2}=5.3\text{ in.}$$

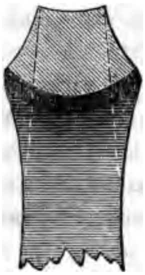
The actual deflection observed ranged from 5 in. to 6 in., the average being 5.35 in., the bars in all instances being uncracked.

Now, let us consider the effect of drawing out on the ultimate deflection of a wrought-iron beam. The first thing to be determined is the amount of that element in a bar under direct tensile strain; and the experiments of Mr. Fairbairn on successive breakages of a bar 1⅜ in. diameter are useful for the purpose, since, the bar being stretched to its full extent by the first strain, the observed amount in following breakages would be entirely due to drawing out. The highest stretch in the second breakage of 36 in. bars was 2 in., and the mean of ten experiments 1.07 in. In 24 in. bars the highest result gave 1½ in. and the average 1.06 in. The greatest stretch in some other bars tested by the same experimenter was 2.04 in.; and in some others, tested by Kirkaldy, .8 in. and 1.3 in.

The depression of some bars 2 diameters long under 37 tons per square inch was found to range from .9 in. to 1.1 in., and of course nearly double those amounts for 4 diameters. It appears, then, that in solid beams of very fibrous iron, the drawing out on the side in tension, together with the depression and bulging out on the opposite side, will amount to some 4 in., and in very ordinary iron to 2 in.

It follows from these conditions that a bar may be doubled up close without cracking, if the diameter do not exceed $\frac{4}{\pi}$ in. $= 1\frac{1}{4}$ in., if of very good iron, or $\frac{2}{\pi}$ in. $= \frac{5}{8}$ in. if of ordinary fibrous iron. The alteration in the form of cross section of a bar thus doubled up is exhibited full size in Fig. 18, which is measured accurately from one so treated. It will be observed that the sectional area and depth are unaltered, whilst the width of the compressed edge is double that of a stretched one.

FIG 18



As the drawing out in a large bar is proved by experiment to be less than that in a small bar, the degree to which a larger bar than 1¼ in. may be doubled up will be rather less than that indicated by the ratios of the depth. Thus, a bar 2½ in. square, if of the same material as the 1¼ in. bar, could be bent only through an angle somewhat less than $\frac{180}{2}$, and so on with larger bars. Should, however, the iron be unusually weak under compression, crumpling up might take place at the compressed edge, and that, of course, would practically reduce the effective depth of the bar, and the angle through which it might be bent would correspond with that due to the diminished depth.

The Admiralty tests for *plates* of the first class are as follows :—1 in. thick plates should admit of bending cold without fracture to angles of 15° with the grain, and 5° across the grain; ½ in. thick plates to angles of 35° and 15°; and ¼ in. plates to angles of 70° and 30°, respectively. Armour-plate bolts made of the best Lowmoor, Bowling, or other highest-class iron, are tested by being doubled cold under the hammer.

We may now pass on to the consideration of the deflection of steel beams. The investigation, as we may anticipate from the nature of the material, will have many things in common with that on wrought-iron beams.

Section 9.

ON THE STIFFNESS OF STEEL.

The rate of elongation of steel under tensile strains varies, within certain limits, inversely as a function of the strength. For average qualities, excluding tool steel, on the one hand, and very soft puddled steel plates of low tensile strength on the other, the mean extension in terms of the length may be taken at .00008 per ton per square inch. Now, it will be remembered, for cast iron we found it to be .00018, and for wrought iron .0001. Looking at these figures, the possibility of the existence of some sort of relationship to the ultimate strength of the materials suggests itself. Thus, taking the ultimate tensile resistance of cast iron, wrought iron, and mild steel at 7 tons, 22 tons, and 35 tons per square inch respectively, it will be apparent that the range of variation in strength is 1 to 5, whilst that of the extension per ton is $2\frac{1}{4}$ to 1, which is very nearly inversely as the square root of the former. Assuming the rate of elongation per ton to be inversely proportional to the square root of the ultimate tensile resistances of the several materials, and taking the value of E for wrought iron = .0001 as the unit, we should have the following values of E for cast iron and steel:

$$\text{Cast iron } E = .0001\sqrt{\tfrac{22}{7}} = .000178$$
$$\text{Steel } E = .0001\sqrt{\tfrac{22}{35}} = .0000795.$$

The hypothesis made appears to hold good, since both these results are practically identical with those derived from direct experiment on the respective materials.

The ultimate elongation of steel is comprised within rather narrower limits than we have found to obtain with wrought iron. The highest rate attained is, probably, that of Krupp's steel tyres, the extension of which sometimes reaches 24 per cent. With Bessemer steel rails the rate varies from 5 to 15 per cent., the average being about the mean of these figures. Puddled bars range from 10 to 20, and puddled plates from 3 to

20, per cent. Of course the higher per-centages are subject to a certain deduction on the score of drawing out; and since the specimens tested, from which these results are deduced, were rather short, that element forms rather an important proportion of the whole.

The next step is to deduce the value of E, from experiments on transverse stress, and we may take first some bars of hematite, 1¾ in. square, tested by Kirkaldy at 25 in. bearings. The average deflection for each 1000lb. is exhibited in the following Table:

Gross Load.	Deflection for each 2000 lb.	
2000	.0014	$E=\frac{4d^3b\delta}{S^3w}$
3000	.0016	
4000	.0014	
5000	.0016	
6000	.0015	
7000	.0014	
8000	.0016	
9000	.0016	
Mean deflection ...	.0015	E=.00008

The following are the results of some experiments on rectangular steel bars of various dimeusions. The thinness of the bars in this instance offered important facilities for accurate measurement of the deflection, which, in each instance, is that due to 1 ton stress applied at the centre of the bearings.

Span.	Width.	Thickness.	Deflection.	$E=\frac{4d^3b\delta}{S^3}$.
In.	In.	In	In.	
35	18	½	$\frac{3}{8}$	.000079
75	24	„	3	.000085
108	48	„	3_2	.000070
29	16	..	¼	.000081
Mean value ...	...	...	...	E=.000079

In the 45 experiments of Mr. Fairbairn, already referred to, the value of E ranged from .000066 to .00009, the mean being .000075. It is evident, then, that for steel, as we have already shown to be the case for cast and wrought iron, the value of E is the same in a beam of rectangular cross section as in a simple bar. We will now take some rail sections, to ascertain if the same conclusion is true for that type.

The deflections of the Metropolitan 84 lb. rail, under successive increments of 2000 lb. stress applied at the centres of 60 in. bearings, are shown in the following Table:

Total Stress.	Deflection for each 2000 lb.	(8 Experiments.)
2.000	.012	
4.000	.012	$E=\frac{24M\delta}{S^3}$.
6.000	.013	
8.000	.019	Hence, since $M=\frac{8.715+11.4}{2}$.
10.000	.020	$E=.000095$.
12.000	.020	
14.000	.019	
16.000	.019	
18.000	.017	
20.000	.020	
22.000	.021	

Mean deflection... .017 for each 2000 lb.

With the same rail reversed, the deflection averaged .016 for each 2000 lb.; hence, $E=.000089$.

Another double-headed 68 lb. rail, 5 in. deep, with a stress of 10 tons at the centre of 42 in. bearings, deflected .065 in.; hence, since the moment of resistance $M=7.8$, we have:

$$E=\frac{24\times 7.8\times 5\text{ in.}\times .065\text{ in.}}{42^3\times 10\text{ tons}}=.000081.$$

The several values of E deduced from the preceding varied experiments on steel beams are .00008, .000079, .000095, .000089, .000075, and .000081; we may, therefore, consider the fair average value to be about .00008, or $\frac{4}{5}$th of that obtained in wrought iron; that is to say, the relative stiffness of steel and iron beams of similar section may be taken as 5 : 4. The deflection of a steel beam of uniform section loaded at the centre to any extent within the elastic limit will, therefore, be given by the formula,

$$\delta=.0000133\frac{S^3\,F}{d}.$$

The *ultimate* deflection of a steel beam will be governed by similar conditions to those affecting wrought-iron beams. As, however, the drawing out at the point of fracture even in the softest descriptions of steel very rarely attains the amount we have found to obtain with tough fibrous iron, the ultimate deflection, or the extent to which a steel *bar* may be doubled up without cracking, will be generally smaller than that of a similar first-class iron bar. Good steel *plates*, on the other hand, sustain without cracking about double the extent of bending endured by first-class iron plates. Thus, the Admiralty test angles for steel plates bent cold, the desirable ultimate tensile resistance of which is specified at 33 tons, and in no case is to exceed 40 tons per square inch, are as follows: 1 in. thick plates 30 degrees with the grain and 20 degrees across it; $\frac{1}{2}$ in. plates 70 degrees and 40 degrees; and $\frac{1}{4}$ in. plates 90 degrees and 70 degrees. The general formula for the ultimate deflection of a steel bar, excluding the element of drawing out, will be as before,

$$\delta=\frac{S^2\ (e+FE)}{12d}.$$

Now some 1¾ in. square bars of hematite steel, tested by Kirkaldy at 25 in. bearings, deflected from 4¾ in. to 7½ in. Hence the ultimate elongation of the same steel in a length of 14 in. being 2½ in., we have $e=.18$; and since $F=50$ tons, we have by the formula,

$$\delta=\frac{25^2\ (.18+.00008\times 50)}{12\times 1.75}=5.4 \text{ in.},$$

which is a fair average of the observed amounts.

The ultimate elongation of the steel in the Metropolitan rails was not ascertained by direct experiment; but from a general consideration of the behaviour of the rails under other tests it may be estimated at 15 per cent. With the flange downwards we found the ultimate strains were 50 tons compression and 38 tons tension per square inch; hence

$$F=\frac{50+38}{2}=44;$$

and since $e=.15$, at 60 in. bearings, the ultimate deflection would be:

$$\delta=\frac{60^2\ (.15+.00008\times 44)}{12\times 4.5 \text{ in.}}=10.3 \text{ in.}$$

It will be remembered that the average result of the observed deflections was 10.5 in.

With the same rail reversed, that is, head downwards, the ultimate deflection was but 5.4 in., or only about half the previous amount. Upon superficial considerations alone this result might appear anomalous, since the mean *moment of* resistance of the cross section, and the deflection within the elastic limit, is the same in both instances. It is, however, easily accounted for. The strains, at the time of fracture, we have found to be 38 tons tension and 29 tons compression; hence, F = 33 tons per square inch. But the maximum compressive strain is practically within the elastic limit, hence the value of *e*, which is the *mean* ultimate elongation and compression, will be reduced to little more than half the former amount, say to 8 per cent. Then, since E = .00008, we have,

$$\delta = \frac{60^2\,(.08 + 33 \times .00008)}{12 \times 4.5} = 5.5 \text{ in.};$$

which is practically identical with that actually obtained.

The ultimate deflection of the rail might have been determined, and a diagram of the deflected rail constructed, by the following method. The radius of the deflection curve at any part of the beam being to the depth, as the latter is to the sum of the extreme elongation and depression of the fibres at the same point, it follows, *e* being the mean sum of the elongation and depression, the radius $r = \frac{d}{2e}$. We have found the value of *e* up to about one half the ultimate resistance of the material to be practically equal to F E, and beyond that limit to increase very nearly as the square of the excess of strain per square inch. In the rail we are now considering we have assumed the maximum value of *e* to be .15; hence the radius of deflection, immediately at the centre of the bearings, would be

$$r = \frac{4.5 \text{ in.}}{2 \times .15} = 15 \text{ in.}$$

The mean radius for the portions within the elastic limit, that is ¼th the span from each bearing, would be

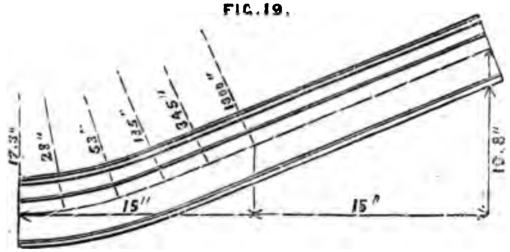
$$r = \frac{4.5 \text{ in.}}{22 \times .00008} = 2550 \text{ in.}$$

Between these points and the centre of the beam the radius would rapidly diminish; taking radii at 2 in., 5 in., 8 in., 11 in., and 14 in., the lengths would be as follows: that at 15 in., or at the centre of the bearings, has already been shown to be 15 in. radius:

15 in. × $15^2 \div 2^2 = 845$ in. radius of deflective curve.
" " $5^2 = 135$ " " "
" " $8^2 = 53$ " " "
" " $11^2 = 28$ " " "
" " $14^2 = 17.3$ " " "

Plotting the deflection curve given by these several radii, the ultimate deflection will be found to be similar to that arrived at by the original method of investigation. The general form of the bent rail exhibited in Fig. 19 is practically identical with that assumed by average rails of the given section under actual test at Kirkaldy's works.

FIG. 19.



Having thus deduced formulæ for the elastic and *ultimate* deflection of cast iron, wrought iron, and steel *beams of* uniform depth and sectional area, subjected to a stress applied at the centre of the bearing, we may, by a few simple considerations, effect the modifications necessary to obtain the deflection of the same or of a different form of beam under any given distribution of the load.

Section 10.

ON THE COMPARATIVE DEFLECTIONS OF DIFFERENT FORMS OF BEAMS.

It has been shown that the deflection of a beam of uniform depth and cross section, sustaining a load at the centre of the span, is to that which would take place if the whole of the extension and compression on the beam were concentrated at the centre, as the contents of a pyramid is to that of the enclosing rectangular prism—that is, as $\frac{1}{3} : 1$. On the simple principles of leverage it was shown that, in the former instance, the deflection would be to the extension and depression of the outer fibres in the semi-beam as the long arm of the bent lever, that is, the half span, is to the other arm represented by the depth of the beam; therefore

$$\delta = \frac{S^2 FE}{2d}.$$

Multiplied by $\frac{1}{3}$rd, this equation gives the deflection for uniform cross sections with the load at the centre; and it is now necessary only to find the corresponding fraction for other cases.

Now if the load, instead of being concentrated at the centre, be uniformly distributed over the beam, it may easily be shown that the strain at any point—the distance, x, from one support—will be to that occurring at the centre of the span, as $Sx-x^2 : \frac{S^2}{4}$, which is one of the equations to a parabola; hence, since the amount of the total deflection contributed by each segment of a beam is, as we have already shown, directly proportional to the strain per square inch at that segment × by the distance from the end of the beam, the solid representing the deflection of the beam under this uniform distribution of the load, instead of being a pyramid, will be, as represented in Fig. 20, a triangle still on plan, but bounded by a parabola in the other direction. The contents of such a solid may be shown to be equal to $\frac{5}{12}$ths of the enclosing prism; that, consequently, will be the fraction to be substituted for the $\frac{1}{3}$rd corresponding to the beam loaded at the centre; hence, the equation for the deflection of a beam of parallel depth and uniform section loaded uniformly becomes,

$$\delta = \frac{5\,S^2\,FE}{24d}.$$

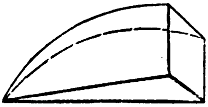


FIG 20

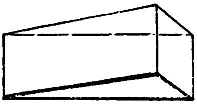


FIG. 21

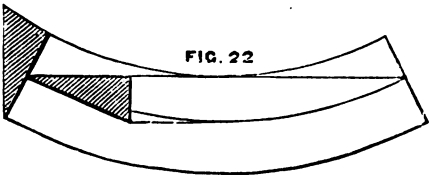


FIG. 22

It appears, then, that under the same strain per square inch, the deflection of an uniformly loaded beam is $\frac{1}{4}$th greater than that of a beam loaded at the centre; or, what amounts to the same thing, with the same load it is $\frac{3}{8}$ths less.

For a beam of uniform *strength*, as well as uniform depth, the strain per square inch being constant, it is obvious the deflection, whether the load be applied at the centre only or distributed, will be represented by the contents of the wedge, Fig. 21, $= \frac{1}{2}$ that of the prism; hence the formula becomes

$$\delta = \frac{S^2 \, FE}{4d}.$$

It follows from this that the deflection of a parallel plate girder, uniformly loaded, reduced in section towards the ends in proportion to the strain, would be $\frac{1}{5}$th greater than it would be were the plates at the centre carried out from end to end.

The same result might have been arrived at on other considerations: thus, the deflection curve of a beam of uniform depth, and with uniform unit strain, being obviously a portion of a circle (Fig. 22), it follows, since the triangles shaded on the diagram are similar, that the deflection, δ, will be to the sum of the extension and compression on the half beam, $e, = F\ E$, as $\frac{1}{4}$th the span is to the depth, hence

$$\delta = \frac{S^2 \, FE}{4d},$$

as before.

Passing on now to the rather more complicated problems involved in the determination of the deflections of beams of uniform strength but varying depth, the first thing is to define the limits within which the actual deflection must be comprised. The condition of uniform unit strain being given, the depth of the beam at any point will obviously be governed by the relative sections of the "web" and "flange" portions of the cross sections; the same elements will also govern the deflection. The two extreme cases will be, first, a beam in which the web portion is small enough to be neglected, and, second, one in which there is no flange portion proper, that is, a beam of uniform thickness.

It is obvious that the greater the relative amount of metal in the flange portions, the smaller will be the average depth, and, consequently, the larger the deflection. In the case of a beam of uniform strength without any web, the outline when the load is applied at the centre will be triangular. For, the flanges being of uniform section throughout, the depths at different points will be directly proportional to the strain, that is, to the distance of the point from the end of the beam.

Now the deflection due to the elongation and compression of the metal at any point, the distance x from the nearest point of support will be to that due to the same change of length occurring at the centre as the distance x, divided by the depth of the beam at that point, is to the half span of the beam divided by the depth at the centre. In the present instance the ratio $\frac{x}{d^1}$ will be constant, and equal to $\frac{S}{2d}$; hence the deflection will be the same as if all the compression and extension occurred at the centre. Under these conditions we have found

$$\delta=\frac{S^2\ FE}{2d};$$

that formula, therefore, will give the deflection of a beam of uniform strength, and little or no web, subjected to a stress applied at the centre.

All practical cases will be included within the limits defined by the preceding equation and the one for a beam of uniform thickness, which will now be deduced.

The strength of a beam of uniform thickness will vary as the square of the depth; hence, for a load at the centre the depth at any point will vary as the square root of the distance from the nearest point of support. The outline of the beam of uniform strength will, therefore, be a pair of parabolas the axes of which correspond with the neutral axis of the beam. The deflection due to any segment of the beam will be, as before, as $\frac{x}{d^1} : \frac{S}{2d}$; hence since $d=\sqrt{x}$, the proportional deflection will also vary, as $\sqrt{x}$, and the mean value of $\sqrt{x}$ will be to $\sqrt{\frac{S}{2}}$ as the area of a parabola is to the enclosing rectangle, that is, as $\frac{2}{3} : 1$. The deflection of the beam in this instance will, therefore, be given by the equation

$$\delta = \frac{S^2 \, FE}{3d}.$$

It is only necessary now to consider the modified deflections of the two preceding types of beams when uniformly loaded.

With a distributed load and little or no web the outline of the girder will be a parabola, the axis of which is at right angles to those in the last instance, and there will consequently be no practical difference in the respective deflections of the two beams. The same formula

$$\delta=\frac{S^2\,FE}{3d}$$

will, therefore, apply.

With a beam of uniform thickness, on the other hand, the deflection will be different to that obtained in either of the preceding cases. The strain at any point of the distance, x, from one pier, y, from the other, will be proportional to xy, and the required depth at the same point will vary as the square root of the strain. The proportional deflection due to each segment of the beam being, as before, proportional to $\frac{x}{d}$, that is, to $\sqrt{\frac{x}{xy}}$, we have the mean value of that fraction in terms of that at the centre of the beam$=\frac{17}{80}$, and consequently the deflection of the beam will be given by the formula

$$\delta=\frac{2\,S^2\,FE}{7d}.$$

Tabulating the preceding results, we find the comparative *stiffness* of independent beams of similar span, depth, and strain per square inch at centre, but with the load and the metal differently distributed, to be as follows:

Weight at centre,	no web proper, and		uniform strain	$=1$
Do.	do.	uniform thickness	do.	$=1\frac{1}{2}$
Weight distributed,		no web proper	do.	$=1\frac{1}{2}$
Do.	do.	uniform thickness	do.	$=1\frac{3}{4}$
Do.	do.	uniform depth	do.	$=2$
Weight at centre		do.	do.	$=2$
Weight distributed		do.	and uniform section	$=2\frac{4}{10}$
Weight at centre		do.	do.	$=3$

No further investigation is necessary to enable the probable deflection of any given section of beam, constructed of either of the materials we have been considering, to be determined with accuracy. It will be necessary, of course, to remember when dealing with beams of unsymmetrical cross section, such as **T** irons, that the value of F will be the mean of the maximum compression and tension occurring on the metal.

PART III.

ON THE STRENGTH AND STIFFNESS OF PERMANENT WAY.

Section 11.

ON THE STRENGTH AND STIFFNESS OF TIMBER.

THE subject of the present section would be but imperfectly elucidated if the problems involved in the consideration of the strength and stiffness of combined beams of iron and wood, upon which the stability of permanent way with longitudinal sleepers so much depends, were omitted from the investigation. Thus, the severest traffic perhaps on any line in the world, that on the original Metropolitan, has been successfully sustained, as far as the absolute strength of the permanent way is concerned, by an iron rail weighing only 60 lb. per yard, the stiffness and strength of which does not amount to $\frac{3}{8}$ths that of the 84 lb. steel rail, adopted on other portions of the same line. The explanation of this apparent anomaly is, that in the former instance the strength of the rail itself is supplemented by that of a longitudinal sleeper 13 in. × $6\frac{1}{2}$ in., to which it is secured by fang bolts, whereas, in the latter case, cross sleepers alone are employed. Whether the maximum bending *stress on continuous* bearing permanent way is in practice greater, or less, than that occurring on cross-sleeper roads, is a question which can only be decided by the indirect results of lengthened practical experience. Some light is thrown upon the subject by the successive modifications in the strength of different systems of permanent way with iron sleepers, suggested by the defects exhibited in the working of each.

One important element in the problem, however, that of the absolute strength and stiffness of different combinations of wood and iron, may be readily arrived at. The first step in the investigation of that question is to determine the strength and stiffness of the longitudinal sleeper *per se*. Ample data are at hand for that purpose in the collected deductions of a host of experimenters on the properties of timber, made years before there were any indications of the appearance of its great rival, iron, on the field.

Unfortunately, most of the careful experiments of Tredgold, Barlow, and other early investigators, were made on small pieces of timber, straight-grained, and free from knots and other defects—a condition favourable, it is true, to the comparison of the results of mathematical investigation with those derived from direct experiment, but, on the other hand, leading to errors of much greater moment in actual practice, since, as every workman knows, a piece of timber uniformly sound throughout can never be reckoned upon. The important per-centage of loss due to the inevitable defects in large scantlings of timber is well indicated in the following experiments:

A piece of English oak, well seasoned and straight-grained, 2 ft. long and 1 in. square, sustained a load equivalent to 8½ cwt., applied at the centre of a bar

12 in. long by 1 in. square. That is, the strength of this piece of oak was little less than half that of a similar bar of cast iron of fair quality. The apparent strain, F, on the extreme fibres of a bar 12 in. long and 1 in. square, subjected to a stress, *w*, being

$$\frac{w \times 12 \text{ in.} \times 6}{4} = 18\,w;$$

in the present instance, it must have amounted to $18 \times 8\frac{1}{2}$ cwt. = 7.6 tons per square inch—a surprisingly high and, as far as practical cases are concerned, a palpably exaggerated result.

Taking a larger scantling of oak, 11 ft. 9 in. long by 8½ in. square, we find the breaking weight reduced to that at the centre of a bar 1 ft. long by 1 in. square, as before, to be 5½ cwt. only, instead of the 8½ cwt. sustained by the last cited specimen. The corresponding value of F, therefore, would be 5 tons per square inch in this instance.

Again, referring to the results of the experiments on the largest scantling of oak beam we are aware of having been subjected to actual test, we find the breaking weight at the centre of the beam, 24 ft. 6 in. long, 12¼ in. deep, and 10⅝ in. wide, was 19,666 lb., which corresponds to a load of 2.7 cwt. at the centre of the 12 in. × 1 in. square bar. This reduced amount shows that the average strength of the timber in this large beam was less than ⅓rd of that in the small selected piece, and we think no further illustration is required to show the necessity of neglecting the majority of experiments on small scantlings of oak when deducing rules for practical application.

We find the same conclusions hold good with reference to Riga, Memel, pitch pine, and other soft woods, the transverse strength of the standard 12 in. × 1 in. square bar of which is, on the basis of experiments on small pieces, stated by various authorities to be from 4 to 5 cwt.; corresponding to a maximum strain on the fibres of from $3\frac{1}{2}$ to $4\frac{1}{2}$ tons per square inch. Deducing results, however, from experiments on beams 15 ft. long by 12 in. square, we find the equivalent breaking weight to be $2\frac{1}{2}$ cwt., or $F=2\frac{1}{4}$ tons per square inch only. Some experiments on creosoted sleepers 14 in. × 7 in., of yellow pine and Memel, are of especial value in the present inquiry. The breaking weight at 5 ft. bearings ranged from 13 to 15 tons; hence the equivalent load on the standard bar would be 1.9 and 2.2 cwt., and the strain $F=1.7$ to 2 tons per square inch. Mr. Brunel also tested a considerable amount of pickled 6 in. × 3 in. scantling, and found the strength of the standard bar to range from 2 cwt. to $3\frac{1}{2}$ cwt. in yellow pine, whilst Memel and red pine offered some 40 per cent. higher resistances.

From a general consideration of the preceding results, it appears that the strength of average samples of sleepers may be computed upon the assumption that the breaking weight at the centre of a bar 1 ft. long by 1 in. square is 2.2 cwt. for pine and 2.75 cwt. for oak. Hence, S being the span in inches, and B W the breaking weight in tons, we have for oak and pine respectively:

$$B\,W=1.65\frac{d^2\,b}{S},$$

$$B\,W=1.32\frac{d^2\,b}{S}.$$

Or the ultimate strength of any beam of rectangular or other form of cross section may be obtained by inserting the values of $F=2$ tons per square inch for pine, or 2⅓ tons per square inch for oak in previous formulæ for the strength of iron and steel beams.

The really practical branch of the inquiry, considered with reference to permanent way with longitudinal sleepers, hinges upon the comparative *stiffness* of the combined rail and sleeper. In respect of the latter member, we find the same exaggerated results given by experiments on small scantlings.

Thus, the mean extension and compression per ton per square inch, in terms of the length, will be given by the equation previously deduced for rectangular iron bars,

$$E=\frac{4\,d^3\,b\delta}{S^3w}.$$

Applying this in the instance of some small bars of oak and pine, we find the value of E to be very variable in different bars of the same timber, and to range from .0012 to .0018. Deducing its value from experiments on the large oak beam already cited, we have the deflection, δ, under a load of 8198 lb. = 2.65 in.; hence, by substitution, we obtain $E=.0022$. Again, taking the large pine beam, we have $E=.003$; and from the following experiments on 14 in. × 7 in. sleepers, tested at 5 ft. bearings, we find the value to range from .0027 to .004.

Weight in Tons.	Deflection. In.	Deflection. In.
2.5	.075	.1
5.5	.163	.25
6.5	.2	.3
7.5	.237	.363
8.5	.263	.425
Mean deflection per ton =	.03	.046

We may, therefore, consider the average value of E for oak and pine to be .0022 and .003 respectively. By inserting these values of E in any of the previous equations for deflection, we may obtain that of any section of beam under any kind of loading.

The ultimate deflection of a timber beam it is impossible to predict; the average amount, however, *before splitting*, appears to be about $\frac{1}{10}$th of the span divided by the depth; but, in most instances, the beam will continue to offer resistance through a considerably greater range of deflection.

The practical difference in the strength and stiffness of oak and pine is less than might have been anticipated. We have found the former to be only $\frac{1}{4}$th stronger, and $\frac{1}{3}$rd stiffer, than the latter. This, perhaps, is partly owing to the greater irregularity in large scantlings of oak.

Having the preceding data, we are in a position to investigate the strength of a combined beam of iron and wood, such as a portion of longitudinal sleeper permanent way.

Section 12.

ON THE STRENGTH AND STIFFNESS OF COMPOUND BEAMS.

The first question suggesting itself is this: Will the rail and sleeper, in a length of ordinary longitudinal sleeper permanent way, act together, or as independent beams? The answer to this obviously is, that it will be entirely dependent upon the efficiency of the fastenings connecting the two members of the compound structure.

The strength and stiffness will, of course, be the minimum when the two members act independently; and although in short pieces prepared for testing, or even in an entire line of railway when newly laid, the structure may be considered, to a certain extent, homogeneous; yet, as every platelayer knows, to his cost, the most unremitting attention will not secure the continuance of that condition. The fastenings, even when fang bolts of the best proportions are employed, will work loose, and, when loose, it is obvious the sleeper and rail must act as independent beams.

It is necessary, therefore, in estimating the stiffness of a longitudinal sleeper road under the worst conditions, to assume that the fastenings do *not* contribute to the vertical stability of the system.

On this hypothesis, M being the moment of cross section of the rail, M^1 that of the sleeper, F the apparent strain in tons per square inch on the extreme fibres of the rail, and F^1 that on the sleeper, we have the weight, W, at the centre of the span, S, equal to,

$$W = \frac{4\,(MF + M^1\,F^1)}{S}.$$

But since the deflection, δ, is necessarily the same both for rail and sleeper, it follows that for an iron rail on a pine sleeper of the depths, d and d^1, respectively, that

$$F : F^1 :: \frac{d}{.0001} : \frac{d^1}{.003}.$$

Hence we have

$$F = \frac{W\,S}{4\left(M + \frac{M^1\,d^1}{30d}\right)}$$

The stiffness of a rail, or any beam, is directly proportional to the product of the moment of resistance of the cross section (M) into the depth (d); hence the stiffness of the combined rail and longitudinal sleeper will, in the same terms, be proportional to $\mathrm{M}\,d+\frac{\mathrm{M}'\,d'}{30}$.

Testing the correctness of this deduction by comparing the results given by it with those obtained by direct experiment on a rail weighing 68 lb. per yard, secured to a 14 in. × 7 in. sleeper, we have $\mathrm{M}=4.5$ for the rail, and $\mathrm{M}=\frac{d^2\,b}{6}=114.5$ for the sleeper; hence, since the stiffness of the combined structure is to that of the rail alone as $\mathrm{M}\,d+\frac{\mathrm{M}'\,d'}{30}:\mathrm{M}\,d$, it follows, the depth of the rail being 3 in., and that of the sleeper 7 in., that the deflection of the rail and sleeper combined, under a given stress, should be the fraction

$$\frac{4.5\times 3\text{ in.}}{4.5\times 3\text{ in.}+\frac{114.5\times 7\text{ in.}}{30}}=\frac{13.5}{40.2}$$

of that obtained by the rail alone under the same conditions.

The deflections of the rail at 5 ft. bearings were as follows:

W.	δ
1.2 tons	.075
2.8 „	.2
3.8 „	.25
4.8 „	.45
Mean deflection	.07 per ton.

It follows, on the hypothesis made in this investigation of the fastenings being inefficient, that the deflection of the rail and sleeper together would at the same span be

$$\delta = \frac{.07 \text{ in.} \times 13.5}{40.2} = .023 \text{ in. per ton.}$$

The actual deflections are given in the following table:

W.	δ
2.5 tons	.025
5.5 "	.1
6.5 "	.1125
7.5 "	.125
8.5 "	.15
9.5 "	.175
10.5 "	.2
11.5 "	.225
12.5 "	.25
15.5 "	.275
Mean deflection	.019 per ton.

The observed deflection, then, was rather smaller than that indicated by the formula. This probably was entirely due to the action of the fastenings, which we have shown should be neglected in the comparison of practical cases. A portion, however, might have been due to the timber being of rather better quality than assumed in the formula. This is one of the many instances in which conclusions, based upon correct theoretical reasoning, are really more reliable than those derived immediately from direct experiments too few in number.

Comparing the stiffness of the preceding permanent way with that of an ordinary double-head rail 5 in. deep, on cross sleepers, the moment of resistance of the rail (M) would have to be equal to $\frac{40.2}{5 \text{ in.}} = 8.04$. Now we have found for a 68 lb. rail of that depth $M = 7.8$, hence the weight of the 5 in. rail on cross sleepers, equivalent in stiffness to the 3 in. rail on longitudinal sleepers, would be about

$$\frac{68 \text{ lb.} \times 8.04}{7.8} = 70 \text{ lb. per yard.}$$

We will now compare the stiffness of the permanent way on the original Metropolitan Railway with that adopted at present on all its extensions. In the former instance the iron rail, 60 lb. to the yard, and 3.45 in. deep, has a moment of resistance $M = 5.9$, whilst that of the longitudinal sleeper 13 in. × $6\frac{1}{2}$ in. $= 92$; hence, the stiffness of the combined structure would be proportional to

$$5.9 \text{ in.} \times 3.45 \text{ in.} + \frac{92 \times 6\frac{1}{2} \text{ in.}}{30} = 40.3,$$

or identical with that of the longitudinal sleeper road last considered.

In the new system of permanent way, with the 84 lb. steel rail on cross sleepers, the depth of the rail is $4\frac{1}{2}$ in., and the moment $M = 10.05$; hence, since we have already shown average steel to be one-fourth stiffer than iron, it follows that the stiffness of this system will be proportional to $1\frac{1}{4} \times 4.5 \text{ in.} \times 10.05 = 56.5$; thus exhibiting an increased stiffness of 40 per cent. over the original permanent way. The actual excess in this particular instance is, however, less, on account of the exceptional softness of the steel.

The deflection within the elastic limit of any section of rail, upon any scantling of longitudinal sleeper, subjected to a stress, W, at the centre of the span, S, will be given by the original equation,

$$\delta = \frac{S^2\,F\,E}{6\,d},$$

when E=.0001 for iron, or .00008 for steel rails, and when

$$F = \frac{W\,S}{4\left(M + \frac{M'\,d'}{30\,d}\right)}$$

for pine, or

$$\frac{W\,S}{4\left(M + \frac{M'\,d'}{22\,d}\right)}$$

for oak sleepers.

Illustrating this formula by its application in the instance of the permanent way, the detailed tests of which are given on pages 133 and 134, we have $M=4.5$, $M'=114.5$, $d=3$ in., $d=7$ in., $S=60$ in., and $E=.0001$; hence the deflection per ton would be:

$$\delta = \frac{60\text{ in.} \times 60\text{ in.} \times 60\text{ in.} \times 1\text{ ton} \times .0001}{6 \times 4 \times 3\text{ in.}\left(4.5 + \frac{114.5 \times 7\text{ in.}}{30 \times 3\text{ in.}}\right)} = .023\text{ in.};$$

which is identical with that previously deduced from the observed deflections of the rail, on the hypothesis that the fastenings were not to be relied upon as increasing the effective stiffness of the road.

From a general consideration of the preceding facts and deductions, it may be concluded that the inherent stiffness of newly laid permanent way, with efficient fastenings, and of the proportions commonly adopted in this country, is the same in continuous bearing longitudinal sleepers road as in those constructed with cross sleepers. But, on the other hand, it is also apparent that the *minimum* stiffness of the road, under the exigencies of actual work, is rather less in the former system than in the latter.

Turning now to the consideration of permanent way constructed entirely of iron, or of a combination of iron and steel, we find that the degree of rigidity possessed by the many different systems, already subjected to the test of practical working, varies considerably from that obtained in timber sleeper roads of the ordinary proportions. Thus, taking two extreme cases, in order to define the limits within which the variations in stiffness will be comprised—namely, the permanent way laid down on a portion of the Rhenish Railway (Fig. 24), and that known as the "Macdonnell" way (Fig. 23), tried on the Bristol and Exeter Railway—we arrive at results which are certainly rather startling. In the former case the permanent way, incontestably the boldest and simplest type yet ventured upon, was little more than a collection of common deck beams (Fig. 24) 11 in. × 4 in., weighing 115 lb. per yard, tied together by 1 in. round bars at intervals of 3 ft. Lateral stability was assumed to be provided by the system of wedging in these deep

FIG. 23.

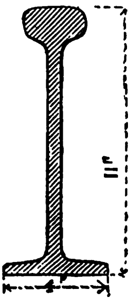
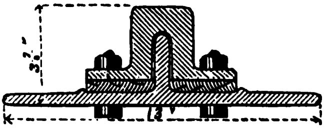


FIG. 24.



rails with fine gravel, in troughs formed in the ballast of broken flints.

The moment of resistance (M) of this section of rail$=34$; hence, since the deflection of a beam of uniform section subjected to a stress at the centre of the bearings is given by the equation

$$\delta=\frac{S^3 \, W \, E}{24 \, M \, d},$$

it follows that the deflection of this rail at 5 ft. bearings for each ton stress applied at the centre, would be .0024 of an inch only. The stiffness of this permanent way is, therefore, nearly ten times that of the ordinary system with longitudinal timber sleepers which it was intended to supersede. We need not be surprised to learn that a differently proportioned and less rigid section was quickly substituted on all subsequent lengths of permanent way laid on this plan.

The other extreme, that of minimum stiffness, is well illustrated by the example selected (Fig. 23). The 68 lb. bridge rail is bolted to a flat longitudinal wrought-iron sleeper weighing about 84 lb. per yard. The section of the latter is rather convenient for the manufacturer than advantageous as a beam liable to be subjected to a considerable amount of bending stress. The intervention of wood packing between the rail and sleeper undoubtedly destroys the homogeneity of the structure, and on that account the conclusion arrived at with respect to the longitudinal timber sleeper road, namely, that the degree of stiffness to be relied upon under actual working is that due to the rail and sleeper acting independently, will also hold good with respect to the *Macdonnell way*.

Now, the deflection of the rail, it will be seen by reference to the tests already tabulated, averaged .07 in. per ton at 5 ft. bearings; and that of the sleeper at the same bearings was .23 in. for a stress of 1.2 tons, and 1.28 in., for a stress of 2.8 tons, giving an average reduced deflection of .33 in. per ton. In the beam made up of these two members, it is evident the proportion of stress sustained by each would be inversely as their deflections; that is, the rail alone would sustain

$$\frac{.33}{.33+07}$$

of the entire stress. It follows that the deflection, for each ton stress applied at the centre of the 5 ft. length of rail and sleeper combined beam, would be

$$\frac{.07\text{ in.}\times .33\text{ in.}}{.33\text{ in.}+.07\text{ in.}}=.059\text{ in.}$$

By actual test the deflections were found to be as follows:

W	δ
2.4 tons	.075 in.
5.6 "	.275 "
6.6 "	.363 "
7.9 "	.5 "

Average deflection = .05 in. per ton.

In the specimens thus tested, the bolts were of course firmly screwed up, which fully accounts for the difference between the results actually obtained, and those previously deduced, on the hypothesis that the fastenings could not be depended upon as an element of vertical strength or stiffness.

Even if we could admit the possibility of the stiffness exhibited by the specimens being obtained in practice, the rigidity of the Macdonnell way would still be considerably less than ½ that of the same rail with timber sleepers in use on the other portions of the same line. Yet this construction did sustain heavy traffic successfully for a number of years; but there can be no doubt the excess of elasticity, or, as some might prefer putting it, the deficiency of rigidity, must have detracted materially from the economy proposed to be effected by the substitution of iron for the perishable material previously in use; since the disturbance of the substructure, and the consequent frequent packing of the sleepers, must have involved much greater labour than would have been necessary had greater vertical stiffness been given, either to the rail or to the iron longitudinal sleeper upon which it rests.

Whilst in the instance of the 11 in. rail on the Rhenish Railway, one could not be surprised to find that the excess of rigidity proved sufficiently detrimental in practice to prevent any but the experimental length being laid down on that system, in the case of the Macdonnell way last referred to, the deficiency of rigidity was so evident, even by theoretical investigation, that it is a matter of surprise the system subsequently adopted, possessing vertical stiffness much more closely approaching that of the timber sleeper road, was not tried at first.

The two systems illustrated, probably define the extreme limits within which the vertical stiffness of any description of permanent way intended to carry ordinary traffic, at present in use, or likely to be ever used, will *be comprised*. And it may well be so, since the one is possessed of upwards of twenty times the rigidity attained by the other.

The earlier rails rolled to Mr. Barlow's well-known "saddle back" pattern were soon found to be unnecessarily heavy. Those tried so far back as 1850, on the North Midland Railway, were 13 in. wide by 5¾ in. high, and weighed 126 lb. per yard. Tested at 52 in. bearings, these rails deflected from .01 in. to .012 in. per ton. Increasing these amounts in the ratio of the square of the span, for the purpose of comparison with experiments on other rails at our standard span of 5 ft., the resulting average deflection would be .015 in. per ton. Permanent way of this type, consequently, would be some 50 per cent. more rigid than the bridge rail with longitudinal sleeper; and, although this amount is insignificant in comparison with the 1000 per cent. excess exhibited by the 11 in. rail, there can be no doubt the substitution of a lighter section was not only perfectly justifiable, but, in all probability, advantageous in every respect.

Section 13.

ON THE PROBABLE MAXIMUM BENDING STRESS OCCURRING ON A RAIL.

Could we take an average of all the systems of permanent way with continuous bearing longitudinal sleepers in use at the present time, and of all those with transverse sleepers, or other analogous supporting mediums, it is more than probable that the vertical stiffness would be found to be the same in each class of road. It was stated, before commencing the investigation of this question of the desirable amount of stiffness for each system, that it could only be definitely and satisfactorily settled by carefully noting the successive modifications suggested by the defects exhibited in different experimental lengths of new systems of permanent way. Upon that class of evidence the preceding conclusion is based.

Viewing the question in a purely theoretical light, it might at first appear that the condition of a rail on a cross sleeper road would be similar to that of a continuous girder supported at fixed intervals, corresponding to the distance apart of the sleepers. If this were true in practice, it would follow, as a corollary, that efficient permanent way might be constructed with any section of rail, no matter how light, provided the cross sleepers were spaced at proportionally small intervals. Indeed, it has been recently urged, by some of the leading authorities on the steel rail question, that the problem of economical permanent way would be solved by the adoption of a light steel rail on a closely sleepered road.

We believe the conclusion thus arrived at is based upon entirely false premises. In fact, if the hypothesis assumed were true, it would follow that a longitudinal sleeper road, instead of requiring, as evidenced by experience, equal strength with a transverse sleeper one, would require no transverse strength proper, but merely sufficient provision to enable the rails to resist the vertical pressure and the crushing action of the wheels.

If we carefully consider the problem presented by a *piece of* permanent way, liable to defective maintenance, and other detrimental contingencies inseparable from the routine of actual work, it will be found that the conditions are those of an elastic beam supported, either at certain points or continuously, upon a partially elastic and unstable foundation, and subject to a rolling load somewhat greater than that carried by the most heavily loaded driving wheel running over the line.

A careful observation of the passage of a heavy locomotive over a piece of permanent way, no matter how perfect it may be, will show that whilst no deflection of the rail between any pair of sleepers, or pair of wheels, is observable, one grand undulation invariably accompanies the movement of the engine. The entire length occupied by the wheel base is depressed temporarily as the load advances, and the original level is practically attained after its transit. In fact, the phenomena are almost the same as those which would be exhibited if the rails were supported upon pontoon sleepers floating in water. It follows from these conditions that—strange as at first sight it may appear—the maximum bending stress occurring on permanent way, either with longitudinal or cross sleepers under the passage of a locomotive of the type at present in use, will be governed, not by the load upon the driving wheels only, but also by the distance apart of each pair of wheels.

This proposition will be rendered clearer by the following illustration: Let us suppose a pair of flanged rails bedded fairly on the natural surface of the soil, which we will, in the present instance, assume to be stiff clay, to be occupied by a train of unloaded waggons. What will be the effects consequent upon loading these waggons? The immediate effect of the application of the first portion of the load will, obviously, be the sinking of the rail into the clay to varying extents at different points of the length; the indentation in the latter being deepest immediately under the wheels, and little, or even nothing, at the intermediate point central between a pair of wheels. This state of affairs would not be long maintained. With the average type of rail and waggon at present in use, after the depression immediately under the wheel exceeded only by $\frac{1}{16}$th of an inch that taking place at the central point between the pair, the increased depth to which the flange would be driven into the clay by further increments of load would be practically the same for all points of the length. The same phenomena would, of course, be exhibited if the rail were founded upon an incompressible rock stratum, provided always a sufficient thickness of elastic material, such as a strip of india-rubber, say one inch thick, intervened between the rail and the rock. In the latter instance, however, the rail would recover its original position upon removal of the load, whilst in the former it, of course, could not do so.

Now, let us take in the same manner a length of ordinary double-headed rail with chairs, the latter being so spaced that one occurs under each wheel, and another between each pair, bedded as before upon a level clay bottom. The result of loading the waggons in this instance would be an initial depression of the chairs immediately under the wheels, to the extent of about $\frac{1}{16}$th of an inch, after which the whole of the chairs would descend simultaneously.

Now, if the entire rail in the instance of the flanged *section, and* the whole of the chairs in that of the double-headed type, would be driven, by the action of the load, uniformly into the hypothetical bed of stiff clay, or would, under the operation of the same agency, compress the elastic supporting medium, represented by the strip of india-rubber, to the same degree at all points of contact, it follows that the reaction of the clay, or the india-rubber, must be the same for each unit of bearing surface. We know, therefore, the distribution of the forces due to the reaction of the supporting medium, and we know also their amount, since it must obviously be equal to the load carried; hence we have all the required data for computing the maximum bending stress which could take effect on any rail, under either of the preceding conditions. For it is evident, without any demonstration, that even if the wheels sank into the clay up to their axle, or the whole train descended any number of feet by the compression of a thick layer of india-rubber, the condition of the rail would be unaltered, provided the rate of sinking were uniform throughout the entire train length.

Now, the actual conditions obtained in an ordinary piece of permanent way are included within the two preceding; the ballast and formation take the place of the clay, and the wooden sleepers that of the india-rubber packing. As we have shown the maximum bending stress to be the same in both, it follows that the same value will obtain also in ordinary permanent way. The obvious corollary to this is, that the amount of bending stress which a given piece of permanent way should be designed to sustain will be unaffected by the degree of instability of the ballast, or elasticity of the sleepers, provided those elements be constant throughout the entire train length; it also is evident that the amount will be the same in a longitudinal as in a cross sleeper road.

Upon superficial consideration, it might appear that the ballast would, in practice, be too stable to admit of the preceding conditions being obtained; but when it is remembered that a depression of but $\frac{1}{16}$th of an inch under the passing wheel is sufficient to make them take effect, the necessity of providing for such a contingency is self-evident. In fact, in our own experience, we have met with one instance in which the bodily sinking of the permanent way of a new line of railway was no less than four inches under the first day's traffic.

In considering this question in its theoretical bearings, it is unnecessary to go into any refinements in order to exhibit the practical conclusions which may be derived from the investigation. For the sake of simplicity, therefore, we shall assume the rails to be continuous throughout, uninterrupted by fished joints of a greater or less degree of efficiency; and we shall consider those rails to be loaded with a row of six-wheeled coupled locomotives, with the wheels spaced at 6 ft. centres. In the cross sleeper road the distance apart of the sleepers will be taken at one-half that of the wheel centres, or 3 ft., and the weight transmitted to the rail by the springs of the heaviest pairs of driving wheels in their most depressed state—that is, of course, when running over a rough road—will be estimated at 9 tons. A lower bending stress cannot justly be taken at present, when dealing with lines worked by the ordinary engines, but *may, at* some future time, should Mr. Fairlie succeed in *imparting to* our locomotive superintendents the philosophical principles regarding locomotion which have been exhibited in his own practice for many years.

Commencing with the cross sleeper road we find two cases presented; one when the driving wheel is over a sleeper (Fig. 25), and the other when it lies between a pair (Fig. 26).

In the first case, the equilibrium of the beam is maintained by the load of 9 tons from each driving wheel acting downwards at 6 ft. intervals, and the reaction of 4½ tons at each sleeper acting upwards at 3 ft. intervals. The condition of the rail, therefore, is that of a continuous beam 6 ft. in span, subjected to a stress of 4½ tons at the centre.

FIG. 25.

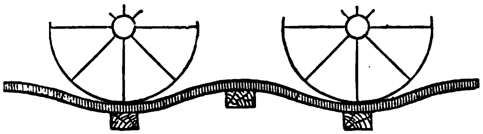


FIG. 26.

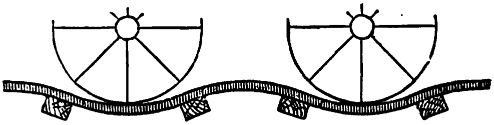
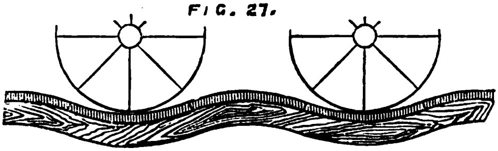


FIG. 27.



With the wheel between a pair of sleepers (Fig. 26) the equilibrium of the rail will be maintained by forces similar to those in the last instance, and the case will be that of a continuous beam 6 ft. in span, carrying two loads of 4½ tons each at the distance of ¼th the span from each end.

Taking now a longitudinal sleeper road (Fig. 27), we have the weight of 9 tons due to the driving wheel acting at a point, and the reaction of sleeper uniformly distributed at the rate of 1½ tons per foot over the 6 ft. span. The strain, consequently, will be similar to that on a continuous beam so loaded.

Now, it will be obvious at once from an inspection of the preceding results, that if the rails were not continuous, but in 6 ft. lengths jointed under each wheel, the bending stress would be the same in all these cases of the problem, and be that due to the load on the driving wheel acting at the centre of a span equal to one half the distance apart of the centres of the wheels. With the proportions adopted, the moment of the weight (m) would, consequently, be equal to 9 tons $\times \frac{72 \text{ in.}}{8} = 81$; and of course that amount divided by the moment of resistance of the cross section of rail employed would give the maximum strain on the metal in tons per square inch. But the rails are not so cut up into 6 ft. lengths. It will, therefore, be desirable now to consider the question of their continuity in order to effect the necessary corrections arising from that condition.

Section 14.

ON THE INFLUENCE OF CONTINUITY.

It may appear a mere truism to state that the work done by a beam, an arch, or any possible agent for carrying a load, must necessarily be always precisely equal to the bending force exerted by the given load; and yet that simple truth is the key to most of the apparently complex problems involved in the determination of the strains on continuous beams.

For, from that condition it at once follows that in a series of continuous girders symmetrically loaded, as our rails are, the strain occurring over each pier together with that at the centre, must be invariably equal to that which would take place at the centre of an independent girder of equal span and similar loading. The real question is, therefore, limited to the determination of the proportions in which this known gross amount will be sustained by the central and end portions of the girder. Two elements are involved in this problem, namely, the distribution of the load and the comparative elasticity of the beam at different points. In the case of the rails both these elements are known; the load in each instance is that due to the reaction of the sleepers, since the work of the piers in an ordinary continuous girder is, on the hypothesis made in this investigation, performed by the wheels of the rolling stock. Again, the rails being of uniform section, the elasticity of the beam will, consequently, be constant throughout. A few simple considerations will enable us to deduce from these known elements the maximum bending moment exerted by the given loads in each case.

In a series of continuous spans equally loaded, no matter whether that load be great or small, vertical lines drawn on the webs of the beams over the piers will, obviously, always remain vertical, and parallel to each other. Such being the case, it follows that—neglecting the infinitesimal correction due to the deflection of the beam—the length of any horizontal fibre in the beam will be constant under every degree of strain, and be equal to the distance apart of the piers, centre to centre. It follows again, as a corollary to this, that the total amount of compression on each fibre must be equal to the total amount of tension on the same. That is to say, the mean compression per square inch multiplied by the length of the fibre in compression must be equal to the mean tension per square inch multiplied by the remaining length of the fibre.

Now, taking the first case of the problem, that with the wheel over a sleeper (Fig. 28), we have the moment (m) at the centre of the independent span, and we have also the moment at any other point, since it will be proportional to the width of the triangle at that point. We know, again, that in this continuous beam, the moment over the pier, together with that at the centre, must be equal to m; hence, in order to obtain a diagram representing the proportional amount of compression and extension occurring on each fibre in terms of that taking place in an independent beam, we have only to draw a line parallel to $a\ b$, at such a *height that* the amount of compression, or extension,

FIG. 28.

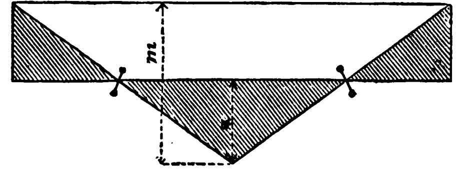


FIG. 29.

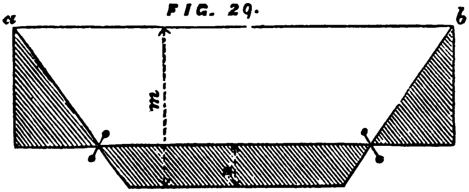
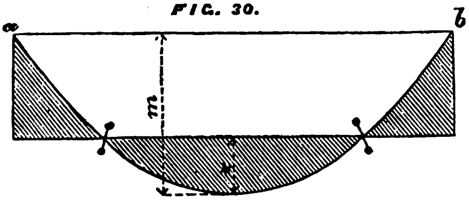


FIG. 30.



as the case may be, represented by the area of the triangles above that line, may be equal to the amount of the tension, or compression, represented by the area of the triangle below the same line. To comply with this condition it is obvious the line must pass through the centre of m. Hence, $x = \frac{m}{2}$; or, in other words, the strain at the centre of this continuous rail will be ½ that previously deduced for the same rail considered as an independent beam, and the strain at the pier—represented by the engine wheel it will be remembered—will be necessarily $\frac{1}{2}$ also. The effective maximum bending moment will, consequently, be equal to the weight on a driving wheel multiplied by $\frac{1}{16}$th of the distance apart of their centres. Hence, for the proportions assumed, it will be

$$\frac{9 \text{ tons} \times 72 \text{ in.}}{16} = 40.5,$$

and this bending moment will occur at the sleepers under the engine wheels, as well as at the intermediate ones. At the former point the top table of the rail will be in compression, and at the latter in tension.

Proceeding now with the investigation of the second case of the problem, when the wheel occupies an intermediate position between a pair of sleepers, we shall find the same reasoning hold good. Constructing a diagram as before (Fig. 29), with the modifications due to the different distributions of the load, it will be seen that, since the areas of the shaded portions above and below the line must be equal, representing, as they do, the proportional amounts of tensile and compressive strain, the only thing required to be ascertained is the value of x, the moment at the centre of the continuous rail, in terms of the known moment m at the centre of the independent rail.

This may be readily deduced from the diagram, for, the areas being necessarily equal, we have $(1-x)^2 = 2x+x^2$; hence, $x=\frac{1}{4}$, that is to say, the strain at the centre of the continuous rail—from sleeper to sleeper—will be $\frac{1}{4}$th that occurring at the centre of the independent rail, and, of course, the strain at the wheel of the engine, taking the place of the pier in ordinary constructions, will be the remaining $\frac{3}{4}$ths. In this instance the maximum bending moment will be equal to the weight on a driving wheel multiplied by $\frac{3}{32}$ of the distance of the wheel centres, or one-half greater than it was when the wheel was over a sleeper. With the 9 ton wheel and 6 ft. centres, the value will be

$$\frac{9 \text{ tons} \times 72 \text{ in.} \times 3}{32} = 60.75.$$

One other case only remains to be investigated—the continuous bearing or longitudinal sleeper road.

Since the reaction of the ballast operates in the same manner as an uniformly distributed load, the diagram for the independent beam will be a portion of a parabola (Fig. 30), the ordinates of which at any point will be proportional to the strain at the same place. As before, the shaded portions above and below the line must be equal. Hence, since the area of a parabola is $\frac{2}{3}$rds that of the enclosing rectangle, we have $\frac{2}{3}x + \sqrt{x} = \frac{1}{3} - x + (\frac{2}{3}x \times \sqrt{x})$, and, by inspection, we obtain $x = \frac{1}{3}$.

The maximum bending moment will, therefore, as in the previous case, take effect under the wheel of the engine; but it will be rather smaller in amount, since it is only $\frac{2}{3}$rds of that at the centre of the independent rail. Generally, it will be equal to the weight on the wheel multiplied by $\frac{1}{12}$th the distance of their centres; and in this particular case it will be,

$$\frac{9 \text{ tons} \times 72 \text{ in.}}{12} = 54.$$

It appears, then, that, on the hypothesis made in the preceding calculations, the maximum bending moment a transverse sleeper road and a longitudinal sleeper road should respectively be calculated to resist will be as follows:

Weight on driving wheel $\times \frac{3}{32} \times$ wheel centres, or for 9 ton wheel and 72 in. centres, $m = 60.75$.

Weight on driving wheel $\times \frac{1}{12} \times$ wheel centres, or for 9 ton wheel and 72 in. centres, $m = 54$.

The reduction in the case of the longitudinal continuous bearing is, therefore, comparatively insignificant. It clearly marks the limit of the reduction in the weight of rail which could be effected by placing the cross sleepers closer together; since it is obvious, if they were even touching each other, the road would still be under the same conditions as a continuous bearing one.

It has been shown that the maximum strain in tons per square inch, occurring on a beam, is equal to the moment of the bending stress divided by the moment of resistance of the cross section; that is, $F = \frac{m}{M}$. Under the heavy traffic to which most railways are now liable, and under all contingencies of working, the value of m has been ascertained to range from 50 to 60 for any description of permanent way; consequently, the ~~maximum~~ strain per square inch on the metal of a rail, in tons per square inch, may be estimated at $\frac{60}{M}$.

In a longitudinal sleeper road the effective value of M has been shown to be equal to $M^1 + \frac{M^2 d^2}{30\, d^1}$; when M^1 = the moment of resistance of the rail, and d^1 the depth of the same; M^2 the moment of resistance, and d^2 *the depth* of the sleeper.

Applying these formulæ to a practical case, that of the longitudinal sleeper permanent way on the Metropolitan Railway already cited, we have $M^1 = 5.9$ in., $M^2 = 92$ in., $d^1 = 3.45$ in., and $d^2 = 6.5$ in.; hence,

$$M = 5.9 + \frac{92 \times 6.5 \text{ in.}}{30 \times 3.45 \text{ in.}} = 11.7;$$

and, since the maximum strain in tons per square inch on the metal has been shown to be equal to $\frac{60}{M}$, it will, in this instance, be $\frac{60}{11.7} = 5\frac{1}{8}$ tons per square inch.

With the solid steel rail on transverse sleepers the value of M for tension is 11.4; hence, the strain will be $\frac{60}{11.4} = 5\frac{1}{4}$ tons per square inch. The rail, as laid with drilled holes, will be liable to an increased strain amounting to some 6 tons per square inch.

It is instructive to observe how closely the relative strength of permanent way, arrived at by a strictly tentative process, approximates to that of other engineering structures. There appears to be some peculiar virtue in a strain of 5 tons per square inch on wrought iron, since, whether the structure be a Britannia Bridge, a steam boiler, or a simple rail, the force the metal has to sustain never varies much from that degree of intensity. This fact should not be forgotten when considerations of economy are prompting the engineer to adopt too light a section of rail; for it is evident that this could not be done without rendering the metal liable to a strain greater than that which practical experience of a varied nature distinctly points out as the useful limits of its resistance.

Section 15.

ON VARIOUS CONDITIONS AFFECTING THE DESIGN OF PERMANENT WAY.

The conclusions arrived at in the preceding section, on the basis of an hypothesis the most unfavourable to the strength of the road which should ever obtain in ordinary fair working, were to the effect that no saving could justly be effected in the weight of the rail, or the scantling of the sleepers, by any modification in the disposition of the supports. This theoretical deduction is amply confirmed by the results of direct experiment cited in previous sections. When a reduced section of rail has been tried, although of course, from the wide range between the desirable working strain of 5 or 6 tons per square inch and the breaking strain, no positive failure has occurred, since the comparatively rapid rate of extension of wrought iron beyond the elastic limit permits of the rail adjusting itself to any unevenness of bearing, yet the excess of strain has evidenced itself in many practical defects, sufficiently important to induce a return to the original heavier section.

On the other hand, when permanent way unnecessarily strong has been attempted, the excess of rigidity has been found to involve other practical inconveniences of equal moment; in the same manner that too great rigidity, either in the springs or the wheels of the *rolling stock*, has been found to militate against the endurance, not only of those members, but of the permanent way itself.

In any attempt, then, to solve the apparently indeterminate problem of really *permanent* way, it must be remembered that at least one element—the desirable degree of vertical stiffness—is well ascertained. The element will be directly proportional to the weight of the rolling stock; and as the driving wheels of the engine are the most heavily loaded, the weight on those wheels should be taken as the datum for computing the required stiffness of the permanent way which has to sustain it. The other conditions affecting the strength of the rail itself are, that the lateral strength should be sufficient to resist the action of the conical tyres, and also of the inevitable blows from the flanges of the wheels; and that the web should be stiff enough to meet the overturning efforts of the same forces. In practice both these desiderata are necessarily fulfilled; since the width of head required for the tread of the wheels, considered in connexion with the convenient distance apart of the sleepers in a cross-sleeper road, or of the transomes in the longitudinal system, is more than sufficient to ensure lateral stability; and the thickness of the web is, from the requirements of the manufacturers, always excessive as far as the mere strength is concerned.

Apart from the required strength of the rail, *per se*, its form will be governed by the strength of the supporting medium immediately in contact with the bearing surface, whether that be iron, hard or soft wood, or ballast simply. In the first instance, when the thrust and pressure on the rail is transmitted to another piece of iron, little or no modification is required in the form of the rail on account of this condition, an illustration of which is afforded by the ordinary double-headed rail supported in chairs. But when wood forms the supporting medium, the area of the bearing surface will obviously be governed by the resistance of that material, and not of the iron rail base. The practical truth of this proposition is demonstrated by the behaviour of flanged rails resting immediately on cross sleepers, as well as by that of the ordinary bridge rail on longitudinal bearers. In exact proportion as the weight of the engines has been increased, so also has it been found necessary to increase the bearing surface of the sleepers. Tredgold found by experiment that a pressure of 1000 lb. per square inch on pine, and 1400 lb. on oak, acting at right angles to the grain of the wood, produced a perceptible and permanent depression. That is to say, the elastic resistance of the wood was overcome by those pressures; hence, in the case of a sleeper, when there are frequent applications and withdrawals of the load, the working strain, if the wood is to remain uninjured, should not exceed one-third of that ultimate elastic resistance, even if we assume the fastenings to be properly drawn up, and no hammering action to ensue. In round numbers, this would give us 3 cwt. per square inch as the working strength of pine, and 4 cwt. as that of oak sleepers.

Now we have assumed the maximum pressure brought upon the rail by the driving wheel of the engine, when the spring is in its most depressed position, to be 9 tons = 180 cwt.; hence in a cross-sleeper road the minimum area of bearing surface at each end of the sleeper should be $\frac{180}{3} = 60$ square inches for pine, and $\frac{180}{4} = 45$ square inches for oak. The width of ordinary cross sleepers being 10 in., it would appear that the minimum width of a flanged rail carrying the heaviest description of traffic should be 6 in., if supported upon pine, and 4½ in. if upon oak sleepers.

When we consider that the pressure is not in practice uniformly distributed over the bearing surface, the weight acting not vertically, but in the direction of the resultant of the vertical and lateral forces, and when we remember also that a certain amount of hammering action is unavoidable, it will be no matter for surprise that, even with the preceding widths of flanges, the sleepers will not in all instances remain uninjured. In fact, the expediency of placing bearing plates between the rails and sleeper, on curves of sharp radius, is plainly evidenced by the results of practical working. It is no uncommon thing to see sound 10 in. × 5 in. sleepers indented to a depth of 1 in. by rails with a 6½ in. flange, and a wider flange than that would be both difficult to roll and extravagant in cost. For these reasons, Continental engineers consider it impolitic to employ any but hard-wood sleepers with the Vignoles rail.

It appears, then, that the width of the flange of a foot rail resting immediately upon cross sleepers is determined by other considerations than those governing the strength of the rail itself; and the same conclusion holds good with reference to the same description of rail, or to the ordinary bridge rail, supported upon longitudinal sleepers.

It is found in practice that about the same width of flange is desirable in the latter case as in the former, since the rail, apparently, does not distribute the weight over a greater area of bearing surface than is afforded by the cross sleepers in the former system. Another reason is that the fibre of the wood, running parallel with the rail, is more unfavourably placed for the duty it has to perform; hence, it has been found advantageous to interpose thin pieces of packing arranged transversely between the rail and sleeper, so as to neutralise, to a certain extent, the splitting action of the rail. The increased resistance which *thin* planks offer to compression is often of great practical importance. In some experiments which came under the notice of the writer, a pressure of 1 ton per square inch, over an area of a circular foot, had to be sustained by some pieces of well-seasoned American oak. With planks 6 in. thick, it was found impossible to put this strain upon the timber, which was invariably crushed into fragments with a much smaller amount. Upon reducing the thickness, however, to 3 in., no signs of distress were evinced.

Should the rail rest immediately upon the ballast, as in Barlow's saddle-back system, the supporting medium will again govern the desirable width of bottom flange. Experience alone can settle the specific amount satisfactorily. On such data it would appear that a minimum width of 1 ft., equivalent to 2 square feet bearing surface per foot run of line, is required to sustain the heavy traffic of railways in the present day. The bearing surface in the old stone block system averaged 2⅔ square *feet per* foot forward; and in the earliest type of cross sleeper roads 1½ square feet, which also was the amount originally provided by De Bergue in his cast-iron sleeper road. On the other hand, in the same class of permanent way by Barlow and others, 2 square feet was the minimum amount allowed. From the results of practice it would appear that the same area is required for continuous bearing sleepers as for other classes of roads.

The office of the transomes in the former system is twofold, namely, to maintain the correct gauge, and also the correct inclination of the running surface of the rails, which should slope inwards at the rate of 1 in 20 about. To do this work; the transomes must possess a certain amount of transverse strength, and we find that, in instances when timber has not been employed, bar iron placed edgeways, T-iron or channel iron, has generally been substituted for the original simple tie bar. Thus, at first Barlow's saddle-back rails were tied together at 10 ft. intervals only, but the defects in some cases were so apparent that triangular timber transomes, notched to the rails, and held by dogs, were afterwards adopted. In the latest system of German iron permanent way, the transomes are of channel iron 4 in. × 1¾ in. × ⅜ in., spaced at 5 ft. intervals, and it is considered by the engineers unadvisable to place them further apart.

Having arrived at the general proportions of the rail and bearer necessary to fulfil the requirements of stiffness and stability, the next *desiderata* to be attended to are, that the parts should be simple and few in number, that there should be few rivets or bolts to work loose, that the strength and stiffness should be maintained as nearly uniform as possible throughout, so as to induce equal wear of the rails, and that the wearing surface of the latter should be formed of the most durable material attainable—a condition easily fulfilled by the adoption of steel.

The necessity of a certain amount of elasticity was very early evidenced by the failure of Hartley's "lathe bed" system, with its rubbed rails and unyielding foundations, and also by that of the portion of the Leeds and Manchester line, laid upon the solid rock, without the intervention of sleepers or ballast. Excess of elasticity, on the other hand, operates in the same way as deficient stability in any other respect. The rails are liable to rock, or to be forced down into the ballast, as in the instances of Greaves's pot sleepers, cited by Captain Sherard Osborne, who reported that in Egypt he had known them in some instances work through 15 in. of ballast down to the solid rock formation.

With timber sleepers, either on the longitudinal or cross sleeper system, sufficient elasticity is attained. In the latter system, however, another important condition —perfect drainage of the substructure—is more completely fulfilled than in the former. When newly laid, there can be no doubt that the longitudinal sleeper road is one of the most perfect. The strength is sufficiently uniform throughout, and an efficient elastic cushion to absorb the blows of the wheels is provided by the wooden sleeper. A consideration of the inherent practical defects of the system, however, leads us to the important question of the durability of timber. In this country the average life of an ordinary sleeper is but seven years; and the statistics of the Belgian lines for the last thirty years show that oak sleepers, costing 5½ francs when new, and worth one-fifth of a franc when old, last, on an average, the extended period of twelve years only.

In a system of framing such as that constituted by the longitudinal sleeper with its transomes, this element of decay is of great practical moment. The timber, alternately wet and dry, is placed under the most disadvantageous conditions, and the result is that the housings of the transomes soon become loose, and packing of some sort is necessary to enable the accurate gauge of the line to be maintained. Independently of the cost of frequent renewals, the great practical inconvenience, and, in many instances, great risk, inseparable from the constant tearing up of different portions of a line, are so obvious, that it is not surprising to find engineers in all countries have for a long period acknowledged the systems of permanent way in ordinary use to be essentially behind the age, and that they have endeavoured to substitute iron for the perishable material hurriedly adopted when the defects of the original stone block road were first evidenced.

All systems of iron permanent way must necessarily belong to one or the other of the two primitive types—continuous bearings and interrupted bearings—the latter including ordinary transverse sleeper roads, in which the special duty of the transome is performed by the centre portion of the sleeper, useless for any other purpose, since in practice the packing seldom extends more than 18 in. on each side of the rail. The former system may be roughly subdivided into two classes—those in which the wearing parts are as light as possible, and capable of easy renewal, and those in which the measure of endurance of the entire structure is practically governed by that of the head of the rail. The former is illustrated by the permanent way adopted on a portion of the Rhenish Railway, where the steel head, secured by bolts to the bearing angle irons, weighs 34 lb. per yard only; and the latter by Barlow's saddle-back and other solid rails. The "Macdonnell" system, and the "suspended girder," or wing angle-irons of W. B. Adams, hold intermediate positions.

The history of Barlow's rail is typical of that of nearly all the other proposed innovations; the evidence as to its success or failure is so contradictory. In some instances the result of experiment would appear to indicate that the form was altogether wrong, since the wedging of the ballast in the interior of the rail tended to split it at the crown; whilst at the same time both the lateral and vertical stability were alike defective on account of the peculiar form of cross section. Notwithstanding these apparently insuperable objections, there can be no doubt that, in many instances, the saddle-back rail has done its work admirably. From a paper on the "Construction of the Track of Railways," read before the Scottish Society of Arts, we learn that the Barlow rail was adopted on the first portion of the Western Railway of Buenos Ayres, as the only method likely to answer the purpose. This railway runs inland over the Pampa, a vast sedimentary plain consisting, according to Mr. Muir, with local exceptions, of a rich black earth of varying thickness, overlying compact clays and marls generally of considerable depth. "Hundreds of square miles may be searched in vain for a porous material, at any workable depth, suitable for ballast. In a six days' journey you may not find a pebble, literally, the size of a pea. . . . . Dryness of bed being absolutely essential for the conservation of the track, as we have not the means of attaining that end by a porous and free material, we are driven to the only alternative of using a comparatively impervious and compact material. This is found to hand in the black loam already spoken of. Care has to be taken to break down the lumps and pack the rails solid underneath. Night dews and occasional rain soon bring the mass together; from its great richness, it is rapidly covered with vegetable growth, and becomes compact and impervious to wet, which drains off the properly rounded surface. While this is in progress, the track is lifted and packed from time to time in the few places where it may have slackened, so that by the time the black soil is drawn together the road is in good line and level."

Under these peculiar conditions none of the presumed inherent defects of the Barlow rail manifested themselves. The gauge was accurately maintained, the stability was perfect, and the smoothness in running was especially noticeable. It is apparent, then, that although no iron permanent way has to any appreciable extent superseded in this country the old-fashioned timber-sleeper road, notwithstanding its irremedial defects, yet in such places as India and Egypt, or Southern America, where timber is scarce, and, if procurable, of no endurance, a very wide field is offered for the economic testing of all possible combinations of wrought-iron or cast-iron supporting media, and possibly some system may at last be evolved which will satisfy the requirements of the most conservative manager of the most heavily worked line in our own country.

The original stone block road possessed some good points, but the rigidity of the structure proved destructive to the weak fastenings; hence the thumping and working of the chairs into the stone—a defect which a layer of interposed felt was perfectly inadequate to remedy. The failure of this road naturally created a prejudice against cast-iron block roads, since the essential difference between the two systems is not necessarily apparent at first sight. Upon more mature consideration, however, it will be evident that a light cast-iron sleeper, securely fastened, metal to metal with the rail, and rising and falling with it, is placed under very different conditions to the massive stone block, the sheer weight of which would generally be sufficient to loosen the imperfect fastenings, even if no play existed between the chair and block.

More than thirty years ago Reynolds introduced his inverted-trough sleepers lined with wood; and some ten years afterwards Greaves's "pot sleeper"—the type of an almost endless variety of more or less ingenious schemes for iron permanent way—was first tested. Under light traffic, in Egypt and elsewhere, the sleeper answered tolerably well; but the long leverage at which the disturbing forces acted, combined with the unstable form of the sleeper, prevented a similar success being attained on English lines. In De Bergue's modification these defects are remedied, the centre of gravity is kept much lower down, and the sleeper is in "stable equilibrium."

Again, the form of the sleeper is such that the weaker portions of the metal are subjected to compressive strains only, which they are well adapted to resist; whereas, in the inverted bowl system the minimum section has to sustain tensile strains, and must necessarily, therefore, be peculiarly liable to fracture. This well-ascertained tendency to fracture on the part of the latter system is partly attributable, also, to the unequal distribution of metal at the projecting lugs for carrying the rails, which are unavoidable sources of weakness to the casting, as they induce irregular contraction, and consequently initial strains at the time of cooling. The laying of a line on De Bergue's system is effected in a most rapid manner, by bolting all the parts together in connexion with one pair of rails, as they rest on seats prepared to receive them on a species of sledge. The length is then lowered into its place, and fished to the preceding one. In this manner it is found in practice that from ½ to ¾ of a mile may be readily laid per diem. A large portion of the Spanish railways are laid on De Bergue's system, and the authorities express themselves perfectly satisfied with the results attained. The renewals and maintenance do not exceed 4 per cent., and in the important matter of first cost, the system bears comparison with the ordinary perishable and otherwise defective timber-sleeper roads. In this country the system has now (1870) been at work on the South-Wales Railway for some ten years, and on the London, Chatham, and Dover Railway for some three years, where, although on made ground, the gauge of the line remains perfect, and no trouble is experienced from noise, vibration, or loosening of the fastenings. For heavy traffic De Bergue employs 66 lb. sleepers, affording a bearing surface of 1.75 square feet per foot run of line. For the Carnatic Railway, Sir Charles Fox is using a 40 lb. steel rail upon 56 lb. sleepers.

The last type of iron permanent way to be referred to—the cross-sleeper road—has not undergone so many modifications as the other systems. It possesses, however, the obvious advantage of fewness of parts; for the duty of the transome is there performed by the sleeper itself. One class of this road, known as the "Système Vautherin," has met with a considerable amount of favour on the Continent. It was first tried in 1864 on the Paris, Lyons, and Mediterranean line; and, after a favourable report, upwards of 100,000 additional sleepers were laid down on that line. In Egypt about the same number are in use, and a considerable quantity are down on many of the French and Belgian lines. The form of the sleeper is that of an inverted trough, with sloping sides and a return flange; the size is 9 in. wide over all, by $2\frac{3}{4}$ in. deep, and the thickness of the wrought iron is $\frac{1}{4}$ in. for the top and flanges, and but $\frac{3}{16}$ in. for the sides. The ballast, it is stated, becomes firmly jammed into the hollow of the sleeper, and assists in maintaining its form. If the strength be found sufficient in practice, there can be little question as to the elasticity; but we do not think any mode of fastening could be devised which would protect such thin metal from the effects of the heavy traffic to be resisted on most English lines. Again, the rapid oxidisation of wrought iron bedded in ballast must speedily effect the destruction of such sleepers in a damp climate such as our own.

Sufficient evidence has been adduced to show that a great demand exists for really efficient iron permanent way. Whether the right system, judged by the standard of economy in first cost, and economy in maintenance, has yet been developed, is the only disputed question. There can be no doubt that iron permanent way may be designed to sustain satisfactorily any class of traffic, and, if the sleeper be kept light and practically homogeneous with the rail, and some elastic medium be interposed between the sleeper and the ballast, it will do its work well on any description of ballast.

PART IV.

ON THE STRENGTH OF COLUMNS.

Section 16.

ON THE ULTIMATE RESISTANCE OF MATERIALS TO COMPRESSION.

THE following investigation must necessarily in many points be closely allied to the preceding one on the transverse strength of beams. Their mutual dependence is self-evident, for no demonstration is required to show the impossibility of foretelling the breaking weight of a beam, failing by compressive strains, unless we know the specific value of the unit resistance to compression, which will vary most sensitively with the length of the unsupported portion of the compression member—usually styled a column when the pressure is induced by an insistant load, and a strut when the thrust is applied in a horizontal or diagonal direction. The formulæ for the transverse strength of beams advanced in previous sections apply to all types of beam in which failure results from excessive tensile strains only. The forms of cross sections referred to were such that it was only necessary then to allude incidentally to the varying resistances of the several materials to compression, but, in many other cases to be met with in practice, the latter element itself would govern the strength of the beam. The unjustifiable extent to which the important difference in the nature of the strain on compression and tension members of a structure is so often ignored, may, no doubt, be partly owing to the fact that in the ordinary run of work—girder bridges, for example—it is unnecessary to entertain the question, because the requirements of the structure generally preclude the possibility of any serious mistake being perpetrated, and in such cases it is quite sufficient to know that the relative areas of the parts in tension and compression, respectively, should be about as 6 to 1 for cast iron, 4 to 5 for wrought iron, and 1 to 2 for wood. But, in any attempt to make an extended application of these proportions to all structures, it is absolutely essential that we should thoroughly appreciate the fact that proper precautions must be taken to ensure the perfect rigidity of all parts under compression, because, whilst the strength of a tension bar depends upon the area of the cross section alone, the strength of a corresponding compression member is, since the material is in a state of unstable equilibrium, ruled as much by the efficiency of the bracing as by the sectional area itself. In many existing arch bridges a deficiency of bracing is very noticeable; in such cases the absolute strength of the structure will be governed by the laws affecting the compressive resistance of long columns.

The determination of the proper strength of a column to carry a given load, being one of the earliest requirements of constructive art, must at some time or another have arrested the attention of every thoughtful man engaged in the design or construction of even the most ordinary class of buildings. It is rather surprising, therefore, that the stock of experiments relating to the subject should, till quite recently, have been so limited and so unworthy the importance of the interests involved. This is the more to be regretted, because the efforts of the many eminent mathematicians who have devoted time and ability to the elucidation of the problem would not have been so often misplaced from the want of proper experimental data by which to test their theories. Conclusions derived from theoretical premises only are often more detrimental than beneficial to the cause they would serve. We have already found that, in estimating the transverse strength of a beam, errors of any extent up to 180 per cent. might result from ignoring the evidence afforded by direct tests, and, for analogous reasons, we must be prepared to find great differences in the strengths of columns as given by various authorities, knowing that careful experiments under the required conditions were, to a great extent, wanting.

The determination even of the fair ultimate resistance to compression of either cast iron, wrought iron, steel, or wood is not so simple an experiment as might at first be imagined. With a tensional strain there is no question when the limit of resistance is reached, since it is evidenced by the tearing asunder of the piece, and then it follows that the strength of any other piece will be simply as its area, assuming of course the pull to be fairly applied, that is, with the resultant passing through the centre of gravity of the cross section. But with a crushing strain, not only has the direction in which weakness will first evidence itself to be determined, which varies greatly with different materials, but it cannot even then be assumed that the resistance will be proportional to the area: in fact, common experience proves to us that, under certain conditions, hardly any limit can be placed to the resistance to compression. The heaviest load we could bring on a thin plank would merely solidify it by reducing the thickness. Thus, the wood packing, sometimes inserted between a double-headed rail and its chair, is found in practice to become so hard ultimately as to eat its way into the rail, to as great an extent as the unprotected cast iron chair itself; and we are told by Mr. Edwin Clark that at one time the greater portion of the weight of one of the Britannia tubes was borne by a short balk of pine 14 in. square, and that although the piece was split in every direction, it continued to support its unprecedented load.

It is obvious, therefore, that, in order to arrive at the fair resistance to crushing of any material, there is a certain limit in the relation of the least side to the height of the specimen experimented upon, which must not be exceeded. Mr. Hodgkinson considered that for cast iron the height of the specimen should not be less than 1½ times its base; and for the following reasons: Finding, by experiment, that fracture usually took place in an oblique direction, he concluded that the resistance must be partly due to shearing, and partly to the frictional resistance of the two wedges into which the material was generally split. Now, if the height of the specimen were too small to admit of the free sliding upon one another of these two wedges, there would be an undue resistance recorded, due to the friction between the ends of the specimen and the surfaces against which it was crushed. If the resistance had been wholly shearing, the line of fracture would, of course, have followed the line of least resistance, which is at an angle of 45° with the axis of the pillar. The effect of the frictional resistance is to reduce that angle. By experiment the actual obliquity appears to be from 32° to 42°; so that a height of 1½ times the base is necessary to admit of the free sliding of the wedges. Adopting, therefore, that necessary precaution in his experiments, Mr. Hodgkinson found the ultimate resistance to crushing of cast iron to vary from 24 to 68 tons per square inch; and he considered that the fair average resistance of good iron might be taken at 49 tons. The latter amount does not differ materially from that generally adopted at the present time in most countries. Till recently a higher ultimate resistance, 10,000 kilos. per centimetre square, about 64 tons per square inch, was admitted in French practice; but all their later authorities have considered it advisable to reduce that amount.

Wrought iron, as we have already shown in former Sections, fails by compression in an entirely different manner to cast iron. Within the limits of elasticity its resistance is easily ascertained, and is found to exceed that of cast iron. It is extremely difficult, however, to determine its ultimate resistance with any precision. Unless the specimen be of sufficient length to fail by crippling, an indefinite amount of bulging out may be the only immediate indication of an excess of load on tough iron. The natural consequence of this is, that little uniformity is to be found in the practice of different countries, and much caution is observable in the application of the results of experiments to practice. Thus, in this country, with the exception of a very few engineers, who consider the resistance to compression and extension to be about equal, from 16 to 18 tons per square inch is taken as the ultimate resistance to crushing of wrought iron, although this is much below the average of recorded experiments on that material. It must be observed, however, that the results of experiments on cells do not justify the adoption of a higher resistance than that usually taken. According to M. Claudel, a resistance of about 25 tons per square inch—6 tons below that given by Rondelet—would appear to be that usually taken in French practice; whilst Dr. Ritter, of Hanover, adopts 19 tons, 6 tons below that given by Wiesbach; and Mr. G. L. Vose cites 21 tons as the usual practice in the United States.

Steel, which, after all, is, as Mr. Bessemer states, nothing more than wrought iron in its most perfect condition, behaves in very much the same manner under pressures of great intensity. The sectional area of the specimen increases *pari passu* with the load, and the ultimate limit of resistance is undefined. For all practical purposes, however, we may assume the comparative *ultimate* resistances of the two materials to be proportional to their *elastic* resistances. On this basis, and from data afforded by experiments carried out at the Arsenal and at Kirkaldy's, the ultimate compressive strength of mild steel may be cited at $1\frac{3}{4}$ times, and that of fine-tempered steel at 3 times, the amount obtained in wrought iron. Hardened steel has in some experiments been subjected to a pressure of nearly 200 tons per square inch.

We might fairly expect to find great discrepancies in the results of different experiments on the compressive strength of wood, conducted independently of each other. Moisture alone, by reducing the lateral adhesion of the fibres, facilitates the process of failure by splitting; hence the ulimate resistance will depend as much upon the degree of seasoning to which the specimen has been subjected, as upon the nature of the material itself. It is, however, difficult to understand how the crushing strength of oak, elm, and other hard woods could ever have been given by the same experimentalist at a less amount than that of such a soft material as pine. This has been done by Rondelet, Rennie, and other good authorities; but it certainly could never have been generally accepted by practical men, accustomed to the use of hard-wood wedges and packing. According to Rondelet, the resistance of wood to crushing varies between the limits of 45 to 54 cwt. per square inch for oak, and 54 to 62 for pine. Tredgold took 36 cwt. for both materials. Rennie states the strength of pine to be 14 cwt., and of elm the absurdly small amount of 11½ cwt. per square inch. Later experiments by Hodgkinson and others place elm at the top of the list with a resistance of 92 cwt., English oak at 2 cwt. per square inch less, and American oak at the same average strength as pine—about 54 cwt. The strength of American oak, however, according to their own experiments, averages 67 cwt., and of elm 71 cwt. per square inch. The strength of all descriptions of timber varies considerably according as the specimen tested is in a wet or dry state.

Having thus briefly reviewed the results of experiments on the ultimate resistance to crushing of short pieces of cast iron, wrought iron, steel, and wood, it is necessary now to consider the influence of length in diminishing that resisting power.

Section 17.

ON CAST-IRON COLUMNS.

It appears somewhat anomalous at first that a compressive force acting along the axis of a column should produce a tension sufficient, if the length exceeds a certain ratio of the diameter, to cause the failure of the column. Assuming the column to be bent laterally to a given extent, the force, applied at the ends, necessary to keep it in that condition is easily determined; but the converse process, namely, to determine the actual stress which would just begin to bend the column, is much involved. The problem has been investigated, however, by Euler, Dr. Young, M. Girard, and many others, but with contradictory results, and a complete theory is yet wanting. The formulæ for the strength of columns given by various authorities are based upon the results of experiments, sometimes direct, but often indirect. Results arrived at by the latter process must be taken with much reservation, as already illustrated by the instance of the transverse strength of a cast-iron bar; and that the same caution must be observed with other materials is shown by the results of Mr. Fairbairn's experiments on wrought T-iron, when the actual strength proved to be about 1½ times the theoretical amount. As might be anticipated, therefore, it is sometimes quite impossible to detect a single point of resemblance in the various formulæ to be met with, many of which give results calculated to mislead seriously. According to Mr. G. L. Vose, the working strength of cast-iron columns 10 in. diameter, and 10 ft. or 20 ft. long, given by six different authorities, is as follows:

	A	B	C	D	E	F
20 ft. long	4,000,000	181,100	370,000	940,000	307,242	300,000 lb.
10 ft. „	8,007,500	204,500	1,442,500	3,640,000	1,170,000	600,000 „

Comparing these results, glaring inconsistencies will be found both in the influence of length and in the amount of stress. We should not on that account ignore all formulæ for the strength of columns. Indirect data of the utmost practical moment is afforded by the formulæ of well-recognised authorities, even though the process of reasoning by which they have been arrived at may not be exhibited. For, at least, it is known that many thousands of columns are doing their work with the proportions derived from those formulæ, and evidence of such a nature it is never justifiable to reject.

In investigations on this subject, it is very generally stated that, if the length exceeds a certain ratio of the diameter, the column will bend, and fail by breaking across at the centre, the point of greatest stress, in precisely the same manner as it would under a lateral strain. Assuming this to be so, since the moment of the force tending to rupture the column is proportional to the product of the load into the deflection, the strength will be inversely as the stiffness; that is, as $\frac{d^4}{l^2}$. But the actual strain on a long column is not wholly of a trans verse nature, since there is an initial compression due to the insistent load uniformly distributed over the cross section, and this consideration is of sufficient importance to modify the above conclusions, slightly in slender columns, but very materially in columns of the proportions usually found in practice. Mr. Hodgkinson, from the results of his very complete series of experiments on the strength of long cast-iron columns with flat ends—the form which will be exclusively considered in the present investigation—was led to the conclusion that the influence of the uniform compression on columns of that material was imperceptible if the length exceeded 35 diameters, but that below that ratio it was essential that it should be included in the calculation. He considered the results of his experiments to be correctly represented by the following well-known formulæ. For columns above 35 diameters, breaking weight in tons, $\text{B.W.} = 44.16 \frac{d^{3.6}}{l^{1.7}}$, ($d$) being the diameter in inches, and (l) the length in feet. For columns below that ratio,

$$\text{B}'\text{W}' = \frac{\text{BW} \times \text{C}}{\text{BW} + \frac{3}{4}\text{C}},$$

C being the ultimate resistance to crushing in tons per square inch—about 49 tons. According to these formulæ, the strength per square inch will not be the same for similar columns—an assumed thing in all other recognised formulæ. The strength per square inch will be less in a certain degree as the absolute diameter of the column is greater. Thus, taking 44 tons = 880 cwt. for the constant, and 49 tons = 980 cwt. as the ultimate resistance to crushing, the breaking weight in cwt. per square inch of a column, 1 in. diameter of a ratio (r), above 35 diameters, will be represented by the formula

$$\text{B.W.}=\frac{1^{1\cdot 6}\times 12^{1\cdot 7}\times 880}{r^{1\cdot 7}\times .7854}=\frac{76560}{r^{1\cdot 7}},$$

and of a column twelve inches diameter by the formula:

$$\text{B.W.}=\frac{12^{1\cdot 6}\times 880}{r^{1\cdot 7}\times .7854}=\frac{59590}{r^{1\cdot 7}}.$$

Below 35 diameters, the breaking weight in cwt. per square inch would be:

$$\text{B}'\text{W}'=\frac{\text{BW}\times 980}{\text{BW}+735},$$

so that, up to 80 diameters, the B.W. per square inch of the 1 in. and 12 in. column respectively would be as follows:

Ratio	5	10	15	20	25	30	35	40	50	60	70	80	
1 in. diameter...	853	661	500	382	298	236	181	144	99	74	56	44	*a*
2 in. diameter...	823	605	440	326	247	184	142	113	77	57	43	34	*b*

Owing to the practical inconvenience of the above method of calculation, involving, as it usually does, two processes, and the use of logarithms, other formulæ, the principle of which will now be explained, are in more general use; they are more consistent to theory, and easier of application.

We have already observed that the strain on a column may be considered as resulting from two distinct forces, one acting along the axis of the column, and the other at right angles to it; the strain per unit of sectional area from the first varying inversely as the area, and from the second inversely as the stiffness of the column. Let W be the load on a column, a its sectional area in square inches, and t_1 the average compression per square inch, which will be the same at every cross section; then $t_1 = \frac{W}{a}$. Now, suppose the column bent laterally to a certain maximum extent, δ, then the moment of the bending force at any portion of the length will be as the deflection at that point; it will, therefore, be a maximum at the centre, where the moment $M = W\delta$; at that point, therefore, the greatest stress will occur, which will be compression or + on the concave, and tension or − on the convex side of the column; and the amount of it will be $t_2 = \frac{n}{I} W\delta$, I being the moment of inertia of the cross section, and n a coefficient depending upon the form of cross section; when symmetrical, $n = \frac{1}{2}d$, d being the diameter. Now, since the moment of the bending force increases more rapidly than the resistance, if the strain exceeds the limits of elasticity, the value of δ will be the ultimate deflection of the column within those limits; that is,

$$\delta = \frac{L^2 m}{8n},$$

m being the fraction representing the increment of length within the elastic limits; that is, the ultimate strain within those limits — the modulus of elasticity. It follows, therefore, that

$$t_2 = \frac{L^2 ma}{8I} t_1,$$

and that the final strain, $t = t_1 + t_2$, will be:

$$t = t_1 \left(1 + \frac{L^2 ma}{8I}\right).$$

Now, for a circular cross section,

$$I=\frac{d^4\,\pi}{64}$$

and

$$a=\frac{d^2\,\pi}{4}$$

therefore,

$$\frac{a}{I}=\frac{16}{d^2}$$

and

$$t=t_1\left(1+2m\frac{l^2}{d^2}\right).$$

Putting $\frac{l}{d}=r$, then

$$t_1-\frac{t}{1+2mr^2}.$$

For cast iron,

$$m=\frac{1}{800},$$

therefore,

$$t^1=\frac{t}{1+\frac{r^2}{400}}.$$

It must not be forgotten, that although the form of the preceding equation is correct, the practical value of the specific results given by it will depend upon the correctness of the constant 400, which has been arrived at by theoretical reasoning from certain assumed data. Thus, we have assumed the load to be uniformly distributed over the ends of the column; but whether this condition really obtains in practice can only be determined by experiment. With square-ended columns, supposing the ends incapable of lateral movement, upon which the whole virtue of the square ends depends, it is quite possible that the beneficial effect of the fixed ends may just counterbalance the detrimental influence of unequal loading; but it is equally possible that it may exceed it, or the reverse; the fact can only be decided empirically. It is rather curious, however, that the value of the constant deduced by Professor Gordon from Mr. Hodgkinson's experiments should be identical with that arrived at on the preceding assumptions; and that the value of *t* in the formula given by him should be about the average ultimate resistance per square inch, as derived from the modulus of rupture and the direct resistance to crushing. Professor Gordon's formula:

$$\text{B.W.} = \frac{36}{1 + \frac{r^2}{400}},$$

will be found to give results agreeing very fairly with those arrived at by the application of Mr. Hodgkinson's formulæ to a column 1 in. diameter. For a 12 in. column the results will be better represented by the formula:

$$\text{B.W.} = \frac{42.4}{1 + \frac{r^2}{250}}.$$

The practical identity of the results, within the limits of 10 to 80 diameters, will be evident on comparison of the following table of the B. W. in cwt. per square inch calculated from these formulæ with the preceding one of the same columns:

Ratio	5	10	15	20	25	30	35	40	50	60	70	80	
1 in. dia. column	678	576	460	360	281	221	177	144	99	72	54	42	*c*
12 in. diameter .	770	605	446	326	242	184	144	114	77	55	41	32	*d*

The formulæ in most general use in France differ little from the two last given. Their authorities, however, have considered it advisable, owing to the great variation in the strength of different sorts of iron, to put the formula in such a shape that the ultimate resistance to crushing of the particular iron proposed to be employed might be substituted in the equation. Thus, C being the ultimate resistance, the formula given by M. Claudel for columns from 5 to 120 diameters is:

$$\text{B.W.} = \frac{\text{C}}{1.45 + .00337r^2},$$

or for columns from 5 to 30 diameters, the more simple formula:

$$\text{B.W.} = \frac{\text{C}}{.68 + .1r}.$$

Substituting C=49 tons, the results for the several ratios would be as follows:

Ratio	5	10	15	20	25	30	35	40	50	60	70	80	
B.W. cwt. ...	639	548	444	350	276	219	176	144	100	73	55	43	e

Dr. August Ritter, of Hanover, whose eminent scientific attainments and practical experience justly constitute him one of the first authorities in Germany, has given a constant .00225 for square columns, which he obtained by the method of investigation already exhibited; it is, therefore, also open to the same exceptions. Reducing for a circular cross section, and taking the ultimate resistance given by Dr. Ritter, that is, 70 kilos. per mil., the formula becomes:

$$\text{B.W.} = \frac{44.4}{1 + .003r^2},$$

and the strengths for the ratios before taken as tabulated below:

Ratio	5	10	15	20	25	30	35	40	50	60	70	80	
B.W. cwt. ..	826	683	530	404	309	240	200	152	104	75	57	44	f

The production of further formulæ would not advance our investigation, although, had we space, the analysis of some of them must, from the curious mixture of elements introduced, have proved at least amusing if uninstructive. We shall, therefore, now proceed to the consideration of wrought-iron columns.

Section 18.

ON WROUGHT-IRON COLUMNS.

It would be difficult to advance more conclusive evidence of the absolute necessity of direct tests than is afforded by the experiments on long wrought-iron columns. We have just found the breaking load of a cast-iron column, 80 diameters long, to be only some 5 per cent. of the ultimate resistance of the material; and we shall hereafter find that the same conclusion holds good, nearly, with wood. Now, whether we consider the comparative strength or the comparative elasticity of wrought iron, it is equally improbable that we should have been led to infer, by analogy, that the strength of a column of that material, 80 diameters long, would amount to 20 per cent. of the ultimate strength—a result, proportionally, four times as large as that obtained in a cast-iron or wood column of similar ratio. A due consideration of the experiments conducted by Mr. Robert Mallet on rolled and forged slabs, showing the high resistance of wrought iron to compression within the elastic limits, and the small change of length resulting therefrom, compared with that exhibited under an equal tensional strain, would undoubtedly prepare us to expect a higher resistance than we otherwise should, knowing how intimately the change of length under compression influences the strain on long columns. It was found by Mr. Mallet that the average resistance to compression without bulging was 25 and 22.7 tons per square inch for forged and rolled slabs respectively; and that the decrement of length caused by those compressions was only about $\frac{1}{10}$ the increment resulting from a tensional strain of equal amount.

It is, however, unnecessary to theorise when we have the valuable series of experiments carried out during the construction of the Britannia Bridge, and recorded by Mr. Edwin Clark, directly applicable to our purpose. The resistance to compression was ascertained; of plates, varying from a length of 10 ft. and a thickness of $\frac{1}{2}$ in., or a ratio of 120 to 1, to a length of $3\frac{3}{4}$ in. and a thickness of 1 in.; and of tubes, from a length of 10 ft. and a diameter of $1\frac{1}{2}$ in., or a ratio of 80 to 1, to a length of 2 ft. 6 in. and a diameter of 6 in., or a ratio of 5 to 1. It was found that, when the length was sufficient to cause failure entirely by flexure, the power of the length, varying from 1.6 to 2.4, averaged as the square; and the power of the thickness, varying from 2.7 to 3.3, averaged as the cube; it was found also that the resistance per square inch of tubes was higher than that of plates of a corresponding ratio; the whole of the facts, so far, agreeing entirely with theory. With square tubes the results did not conform in the same satisfactory manner, owing to the fact that the thickness of the metal, although unimportant theoretically, is practically very influential in determining the ultimate resistance per square inch. It was proved generally, however, that the ultimate resistance, about 12 tons per square inch, was obtained when the thickness of the metal was about $\frac{1}{30}$ the side, and the side $\frac{1}{30}$ the length; and that a prejudicial effect followed the decrease of either of these ratios at the cost of a corresponding increase on the other. The experiments on gas-tubes gave higher results, the metal being in a better form to resist distortion; the ultimate resistance of short lengths, however, was not so great as that obtained with plates, the respective amounts being about 18½ and 23½ tons per square inch; but at the highest ratios the tubes had an advantage, amounting, when the ratio was 80 to 1, to about 50 per cent. The general results of all the experiments, reduced for a circular cross section, are well represented by the formula:

$$\text{B.W.} = \frac{18}{1 + \frac{r^2}{2000}},$$

and the following table shows the breaking weights in cwt. per square inch so deduced.

Ratio ...	5	10	15	20	25	30	35	40	50	60	70	80	
B.W. cwt.	355	343	323	300	274	248	223	200	160	128	104	85	g

*Prof*essor Rankine gives the formula:

$$\text{B.W.}\frac{36000}{1+\frac{r^2}{36000\,c^2}}$$

for the breaking load of a wrought-iron column in lbs. For a circular cross section

$$c^2=\frac{d^2}{16},$$

the formula, therefore, becomes:

$$\text{B.W.}=\frac{36000}{1+\frac{r^2}{2250}}.$$

On the other hand, Professor Gordon's formula is:

$$\text{B.W.}=\frac{16}{1+\frac{r^2}{3000}}$$

for the breaking weight in tons. An application of these respective formulæ gives the breaking weight of a round wrought-iron column in cwt. per square inch, at the following amounts:

Ratio ...	5	10	15	20	25	30	35	40	50	60	70	80	
Rankine	318	308	292	273	252	229	208	188	152	123	101	83	*h*
Gordon.	317	310	297	282	265	255	227	208	175	145	122	102	*i*

The theoretical strength of wrought-iron columns has been investigated by Dr. Ritter; but, as might have been predicted from a due consideration of Mr. Mallet's experiments on the elastic resistance of that material to compression, his results show a very considerable underestimate of the actual strength proved to exist by direct experiment. The constant .001125 arrived at by Dr.

Ritter, is just ½ that given by him for cast-iron columns, whereas the formulæ we have given which are deduced from experiment, show the real value of the constant for wrought iron to be certainly not more than ⅕ that obtained with cast-iron columns. Taking an ultimate resistance of 30 kilos. per mil., as given by Dr. Ritter, and reducing for a round column, his formula becomes:

$$\text{B.W.} = \frac{19}{1 + .0015r^2},$$

and the results of the applications of it are as follows:

Ratio..	5	10	15	20	25	30	35	40	50	60	70	80	
B.W...	366	330	283	237	196	161	134	111	80	59	45	35	*j*

The formulæ given by M. Claudel are similar to those already quoted from him for cast-iron columns. The constant for wrought iron is ⅓ that given for cast, but a higher ultimate resistance—8000 kilos. per cent.—is taken than in either of the other formulæ we have given. For columns from 10 to 180 diameters his formula is:

$$\text{B.W.} = \frac{25.4}{1.55 \times .0005r^2},$$

or from 5 to 30 diameters the simpler one:

$$\text{B.W.} = \frac{25.4}{.85 \times .04r}.$$

The following table shows the corresponding breaking weights in cwt. per square inch:

Ratio..	5	10	15	20	25	30	35	40	50	60	70	80	
B.W...	325	317	300	290	272	254	235	216	181	151	127	107	*k*

Section 19.

ON WOOD COLUMNS.

The present almost universal adoption of cast or wrought iron to the exclusion of wood, wherever a heavy strain has to be encountered, is quite modern in its origin. Little more than twenty years intervened between the first substitution of an iron girder and column for the universal wooden brestsummer and post, in a shop front in the Strand, and the erection of the Exhibition Building in Hyde Park. The amount of adverse criticism which followed the first-mentioned innovation proves that even at that period it was in advance of the times. Now, since expensive experimental investigations seldom take place, unless the absolute necessity of them is apparent, we need not be surprised to find, that whilst all the experiments on iron columns to which we have had occasion to refer are of recent date, nearly all the experiments on wooden columns belong almost to the pre-engineering epoch.

Mr. Tredgold, the authority usually consulted on matters connected with "carpentry," derives his rule for the strength of columns from the experiments of MM. Lamandé and Girard. The specimens subjected to test in these instances were pieces of oak, varying in size from 8 ft. 6 in. × 4¼ in. square to 2 ft. 3 in. × 2⅛ in. square. The successive increments of load, and the corresponding bendings of the columns, if any, were carefully noted. As Mr. Tredgold, however, considered that the investigation should be confined to the load producing the first degree of flexure, the only use he makes of the results is to determine the value of a constant,

$$e=\frac{d^4\,\delta}{l\,w},$$

d being the side of the square in inches, l the length in feet, w the load in pounds, and δ the corresponding deflection in inches. According to Lamandé, the value of e ranges from .00015 to .00125, and, according to Girard, from .00016 to .00093. Tredgold considered, therefore, that by taking $e=.0015$, the column would be strong enough for use, as in good timber it would seldom be loaded with more than one-fourth the weight producing sensible flexure, if calculated by his rule, which, for square columns, is

$$d^2=l\sqrt[2]{w\,e}.$$

Now, since it is evident that the load should never exceed a certain fraction of that which would crush a short length of the column, the result given by the preceding formula will be excessive, unless the length exceed a certain ratio of the diameter. To determine that ratio we have, taking the ultimate working strength at 1000 lb. per square inch, as given by Tredgold,

$$1000d^2=\frac{d^4}{l^2.0015},$$

therefore,

$$l^2=\frac{d^4}{1000d^2.0015},$$

and

$$l=d\sqrt[2]{\frac{1}{1000\times 0015}}=.818.$$

As the length is taken in feet, the equivalent ratio will be rather less than 10 to 1. According to Tredgold, therefore, the working strength of a square column below that ratio will be constant, and be equal to 9 cwt. per square inch, and for any higher ratio will be the amount shown in the following table, which is calculated from the formula—working load in cwt. per square inch

$$=\frac{d^4 12^2}{l^2 .0015 d^2 112}=\frac{857}{r^2}.$$

Ratio.....	5	10	15	20	25	30	35	40	50	60	70	80	
W.L......	9	8.57	3.8	2.14	1.37	.95	.7	.53	.34	.24	.18	.13	l.

It may be observed that, in the above experiments, the ultimate loads borne by the columns were found to be about double those producing the first degree of flexure.

A large proportion of the numerous formulæ for the strength of wood columns are mere modifications of the one given by Professor Barlow, which is not based upon the results of direct experiment, but upon Dr. Young's investigations on the weight a column should *theoretically* be able to support without sensible flexure. Dr. Young's formula is,

$$W=\frac{.8225 d^2 g}{l^2},$$

W being the load in pounds, l and d the length and thickness, both in inches, and g the weight of the modulus corresponding to the section, and equal, therefore, to $d^2 \times$ modulus of elasticity (m), consequently,

$$W=\frac{.8225 d^4 m}{l^2}.$$

Professor Barlow, in his formula, substitutes E, a reduced value of m,

$$E=\frac{3m}{12^3\times 32};$$

therefore, taking the length in feet,

$$W=\frac{.8225d^4 E 18432}{l^2\,12^2}=\frac{80\,Ed^4}{l^2}$$

and,

$$d^4=\frac{Wl^2}{80E},$$

which is the rule given by Professor Barlow; accompanied, however, by the necessary reservation that the results given by it are too small for practice, since the smallest deviation of the pressure from the axis of the column would overpower its resistance. Taking $E=96$ as deduced by Professor Barlow from his experiments, and the working load in cwt. per square inch (WL) at ¼ the amount given by the formula, then,

$$WL=\frac{80\times 96d^4 12^2}{4l^2 112 d^2}=\frac{2465}{l^2},$$

and the ratio below which the ultimate working strength of 9 cwt. per square inch remains constant, is,

$$\sqrt{\frac{2465}{9}}=17.$$

For the other ratios the results are as follows:

Ratio.....	5	10	15	20	25	30	35	40	50	60	70	80	
W.L......	9	9	9	6.16	4	2.74	2	1.54	.98	.68	.5	.38	m

Professor Hodgkinson made a few experiments on pieces of wood varying in length from 30 to 45 diameters, and gave the following formula for the breaking weight in lbs.: $BW=3{,}000{,}000\frac{d^4}{l^2}$, or in cwt. per square inch:

$$\text{B. W.}=\frac{26785}{r^2}.$$

Taking 53½ cwt. per square inch as the crushing load, the ratio below which the strength according to this rule would be constant is,

$$\sqrt[2]{\frac{26785}{53.5}}=22.$$

The constant 26785 is, therefore, obviously too large; it is, however, somewhat corrected by the adoption of the high factor of safety—10—recommended by Mr. Hodgkinson. For the purpose of comparison with Tredgold's and Barlow's deductions, we give the working load (W.L.) in addition to the usual table:

Ratio..	5	10	15	20	25	30	35	40	50	60	70	80	
B. W..	53.5	53.5	53.5	53.5	42.8	29.8	21.8	16.7	10.7	7.4	5.4	4.2	*n*
W.L...	5.35	5.35	5.35	5.35	4.28	29.8	2.18	1.67	1.07	.74	.54	.42	*o*

Professor Rankine and Dr. Ritter have given formulæ for wooden columns of the same class as those given by them for iron ones. In this instance, as in the previous one, Professor Rankine's rule is based upon experiment, and Dr. Ritter's upon theoretical considerations only, Reducing both formulæ to the same unit of measurement—cwt. per square inch—the breaking weight, according to Rankine, is,

$$\text{B. W.}=\frac{27}{1+\frac{r^2}{250}},$$

and, according to Ritter,

$$\text{B. W.}=\frac{53.5}{1+0027r^2}.$$

Adopting the factors of safety recommended by the respective authorities, the breaking weights (B.W.) and working loads (W.L.) will be as follows:

Ratio............		5	10	15	20	25	30	35	40	50	60	70	80
Rank.	B.W.	24.6	19.3	14.2	10.4	7.7	5.9	4.57	3.64	2.45	1.76	1.36	1.02
	W.L.	4.9	3.9	2.9	2.1	1.8	1.2	.92	.73	.5	.36	.28	.21
Ritter	B.W.	50.1	42.1	33.3	25.7	20	16	12.4	10	6.9	5	3.8	2.9
	W.L.	7.1	6	4.8	3.7	2.9	2.3	1.8	1.4	1	.7	.54	.4

M. Rondelet says, that if we take the resistance of a cube of wood as the unit, the reduced strength corresponding to the different ratios will be as follows:

Ratio..................	1	12	24	36	48	60	72
	1	$\frac{5}{6}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{12}$	$\frac{1}{24}$

Rectifying the curve, and taking 53½ cwt. per square inch as the unit, we obtain the following amounts for the B.W. and W.L. (as recommended by M. Morin) for the ratios adopted in the previous tables:

Ratio..	5	10	15	20	25	30	35	40	50	60	70	80	
B. W..	47.8	42	36.4	31	26.3	21.7	17.7	14	8.6	4.8	2.5	1.4	*t*
W.L...	6.8	6	5.2	4.4	3.8	3.1	2.5	2	1.2	.7	.35	.2	*w*

For reasons that will be evidenced hereafter, it will be more convenient to defer, for the present, the consideration of steel columns. We proceed, therefore, at once to collate the evidence already before us relating to the strength of cast iron, wrought iron, and wood columns.

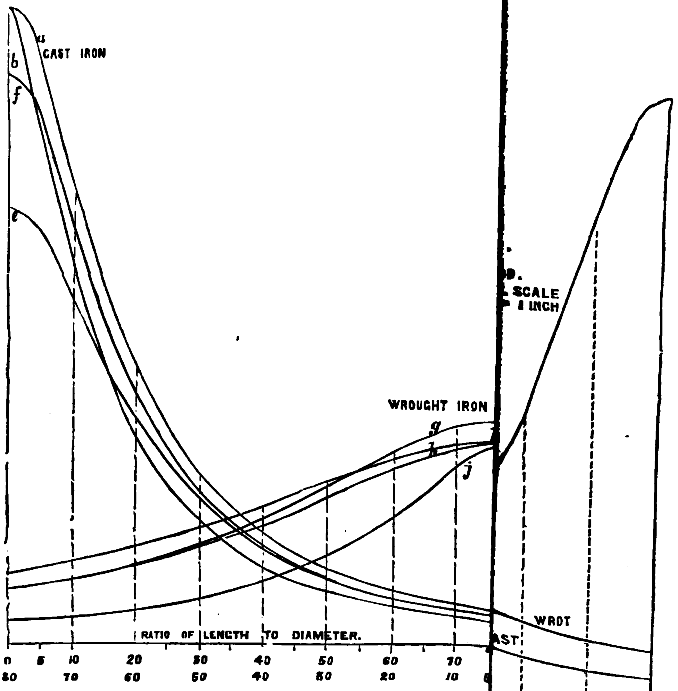


DIAGRAM No. 1.

SCALE 800 CWTS PER SQ: INCH = 1 INCH.

ce page 197.

Section 20.

DEDUCTION OF FORMULÆ FOR CAST IRON, WROUGHT IRON, AND WOOD COLUMNS.

Having now examined all our witnesses, a comprehensive view of the evidence will be best attained, and the process of summing up most facilitated by giving the several tables, resulting from the application of the various recognised formulæ, in the form of a diagram, since by this graphical representation a much clearer estimate will be formed of their functional differences than could possibly be obtained by a mere inspection of the figures in the tables themselves. In order to save space, we have made diagram No. 1 serve for both cast and wrought-iron columns. As the curves are plotted to a vertical scale of 300 cwt. per square inch = 1 inch, and the horizontal line is divided into equal parts representing successive ratios up to 80 diameters, increasing from left to right for cast and from right to left for wrought-iron columns, of course the ordinate of any curve, measured from the point on the base line corresponding to the given ratio, represents the breaking weight in cwt. per square inch, as determined by the formulæ from which the particular curve was derived.

In diagram No. 2, referring to wooden columns, we have combined the several tables of the ultimate and working loads per square inch. The curves representing the ultimate strengths are plotted to a vertical scale of

15 cwt. to an inch, the ratios increasing from left to right; those representing the working loads are drawn to a scale five times larger, and the ratios run in the reverse direction.

With the diagram before us, we can hardly fail being struck with the contradictory nature of the evidence relating to the strength of wooden columns, so plainly is it indicated by the irregular form of No. 2 diagram; in comparison, the consistency of the results obtained for iron columns is very marked. These facts should be allowed due weight when determining the desirable factors of safety. It is our duty now to devise convenient rules fairly representing the balance of authority in each instance.

First, taking the diagram for cast-iron columns, we find the mean heights at the two ends to be 820 cwt. and 40 cwt. respectively. We have, therefore,

$$\text{B. W.} = \frac{820}{1+\frac{r^2}{x}},$$

consequently,

$$x = \frac{r^2 \, \text{B W}}{820 - \text{B W}}$$

Since B W = 40 when $r = 80$ we have,

$$x = \frac{80^2 \times 40}{820-40} = 330,$$

and,

$$\text{B. W.} = \frac{820}{1+\frac{r^2}{330}},$$

or, in round numbers,

$$\text{B. W.} = \frac{270{,}000}{330+r^2}.$$

Proceeding in the same manner with the diagram for wrought-iron columns, we find the respective heights to be 340 cwt. and 81 cwt.; therefore,

$$x=\frac{80^2\times 81}{340-81}=2000,$$

and,

$$\text{B. W.}=\frac{340}{1+\frac{r^2}{2000}},$$

or, in round numbers,

$$\text{B. W.}=\frac{680{,}000}{2000+r^2}$$

With diagram No. 2 it will be necessary to adopt a somewhat different method. Taking first the curves for the breaking weights, we find the average heights at ratios 5, 40, and 80, to be respectively 44, 11, and 2.4 cwt.; then from the curves for the working loads we obtain corresponding average heights of 7, 1, and 3 cwt. It appears, therefore, that the factor of safety varies from $\frac{44}{7}=6\frac{1}{3}$ to $\frac{2.4}{.3}=8$. Assuming 7 to be the factor at the ratio of 40, and converting the working loads into the equivalent breaking weights, by multiplying the several amounts by their respective factors, we get 44, 7, and 2.4 cwt. Now, taking the mean between these amounts and those obtained directly from the diagram, we obtain 44 cwt., 9 cwt., and 2.4 cwt. per square inch, as the breaking weights for the respective columns.

Now,

$$\text{B. W.}=\frac{C}{1+\frac{r^2}{x}},$$

therefore,

$$C=B.W.\left(1+\frac{r^2}{x}\right)=B'W'\left(1+\frac{r_1^2}{x}\right)$$

and,

$$x=\frac{(B.W.r^2-B'W'r_1^2)}{B'W'-BW}.$$

When $r=40$, we have B. W. $=9$, and when $r_1=5$, then B′ W′ $=44$, therefore,

$$x=\frac{(9\times1600)-(44\times25)}{44-9}=380,$$

and,

$$C=\frac{44(380+25)}{380}=47.$$

Again, when $r=80$ B.W. $=2.4$, and when $r_1=5$ B′ W′ $=44$, therefore,

$$x=\frac{14360}{41.6}=343,$$

and,

$$C=\frac{16192}{343}=47.5.$$

Also, when $r=80$ B. W. $=2.4$, and when $r_1=40$ B′ W′ $=9$, therefore,

$$x=\frac{960}{6.6}=146,$$

and,

$$C=\frac{15714}{146}=108.$$

We find, therefore, the value of C varies from 47 to 108, and of x from 380 to 108. Assuming the mean values to be 48 and 350, we obtain, for the ratios 5, 40, and 80, the amounts 45, 8.7, and 2.5 cwt., which approximate sufficiently near to the actual mean (44, 9, and 2.4) to justify us in adopting those constants. The formula, therefore, becomes:

$$\text{B. W.}=\frac{48}{1+\frac{r^2}{350}},$$

or:

$$\text{B. W.}=\frac{17000}{350+r^2}.$$

The result of our investigations so far is represented by the following formulæ and table, and by diagram No. 3:

$$\left.\begin{array}{l}\text{Round}\\ \text{Columns}\end{array}\right\{\begin{array}{llll}\text{Cast iron B. W. in cwt. per square inch} & & & =\dfrac{270{,}000}{330+r^2}\\ \text{Wrought iron} & \text{do.} & \text{do.} & =\dfrac{680{,}000}{2000+r^2}\end{array}$$

$$\begin{array}{llll}\text{Square Columns, wood} & \text{do.} & \text{do.} & =\dfrac{17{,}000}{350+r^2}\end{array}$$

Ratio............	0	5	10	15	20	25	30	35	40	50	60	70	80
B. W. Cast.......	820	761	628	486	370	283	219	173	140	96	69	51	40
Wrought	310	335	324	306	284	259	235	211	188	151	121	99	81
Wood....	48	45.3	37.7	29.5	22.6	17.4	13.6	10.8	8.7	5.9	4.3	3.2	2.5

Concerning the rule for wooden columns, it must be observed that hitherto we have made no distinction between hard wood and pine. Now, although hard wood has a theoretical advantage at all ratios, it is doubtful whether in practice it extends beyond the ratio of 15 diameters. Assuming this to be so, as our resultant formula gives the strength of pine columns (for, although partly derived from experiments on oak, the lengths experimented on averaged more than 15 diameters), and as the ultimate resistance to crushing of hard wood is about 30 cwt. per square inch greater than that of pine, the breaking weight B′ W′ of an oak or elm post, below the ratio 15, will be represented by the formula,

$$B' W' = \frac{B W 45}{30 + r}.$$

The desirable factors of safety for the three materials, when of average quality, appear to be as follows: For cast iron, from 5 to 7; for wrought iron, from 4 to 5; and for wood, from 7 to 10; the higher amounts to be adopted when the live load is considerable. Although, according to the principles of dynamics, a suddenly applied load involves double the deflection, and consequently double the strain, due to the mere dead weight alone, experiments on the comparative deflections of bridges under dead and moving loads show that such a difference does not obtain in practice.

Section 21.

ON STEEL COLUMNS.

We have for several reasons deferred the consideration of the strength of steel columns till the present stage of the investigation. In the first place, the stock of experiments relating to the subject is not sufficiently comprehensive to justify us in throwing over the valuable *data* afforded by the far more extensive series of experiments on the analogous materials, cast and wrought iron, when considering the probable strength of a steel column. In the second place, the results of those experiments which have been made on the compressive resistance of steel, present an embarrassing want of uniformity.

When devising convenient rules for the strength of steel columns it will, therefore, be necessary always to keep in mind the special conditions induced by the ascertained irregularity in the properties of the metal. In order to appreciate thoroughly the relative practical importance of the several variable conditions involved, it will be well to consider the question briefly in its theoretical bearings.

The formulæ advanced in previous Sections refer in all instances to columns with flat ends; that is to say, to columns secured in such a manner that the planes of the two ends must necessarily remain parallel to each other under all degrees of flexure. The strength of a long column thus fixed is greater than that of one merely loosely butted at each end, in the same proportion as the strength of a continuous girder is greater than that of an independent one, and for similar reasons. It may readily be proved that the points of contrary flexure in a continuous girder of uniform section are .211 of the span distant from each support; the length of the central portion, consequently, will be .578 of the span. It follows, from the preceding considerations, that a long column fixed at each end may be conceived as made up of a central column .578 of the total length (L), butting loosely against the ends of two half columns, fixed at the remote ends. Since the two halves are merely portions of a column .422 L in length, it will be necessary to deal with the longer central segment only, in the following theoretical investigation of the resistance of the entire column.

Given a thin elastic bar of steel of the length .578 L, bent, to any extent within the elastic limit, by a force applied laterally, and maintained in that condition by a tie rod connecting the opposite ends of the arched bar; then, it would obviously follow, that upon the withdrawal of the lateral pressure, a tensional strain of certain intensity would be induced upon the tie bar. It is equally obvious that, if the bar were set up on end, the tie might be replaced by an insistent weight, equal in amount to the original pull upon that member.

It follows as a corollary to the preceding proposition, that the maximum tension induced upon the tie, by the elastic reaction of the bent bar, will represent the limiting weight that bar would be capable of sustaining as a column. With the aid of the formulæ for the deflection of beams, advanced in previous Sections, we may readily deduce the required value of the elastic reaction. For we have given an elastic column bent to a certain extent, and we can at once obtain the amount of the stress which, applied laterally, would deflect the bar an equal degree. From this data we can easily derive the intensity of the pull upon the tie bar connecting the opposite ends of the arched bar, since it will be practically identical with the thrust of an arch of non-elastic material, of which the rise is represented by the deflection of the bar, and the load by the lateral stress required to induce that deflection.

Now the moment of the bending force occurring on different portions of a bent column, from the action of the insistent load, will be proportional to the ordinates of the deflection curve at the same points. This curve *will differ* little from a parabola, hence we may assume the moments on the bent column to be the same as if it were subjected to a *distributed* lateral pressure of certain uniform intensity. But it has already been shown that the deflection of a round bar, of the length .578 L, under a distributed load, will be given by the equation,

$$\delta=\frac{5FE(.578L)^2}{24d}.$$

In which the weight (W) being distributed, the value of F will be:

$$F=\frac{4W(.578L)}{\pi d^3}.$$

But the thrust of a non-elastic arch of the rise δ, under the distributed load W, would be,

$$T=\frac{W(.578L)}{8\delta}.$$

Hence, by substitution, we obtain,

$$T=\frac{9\pi d^4}{20L^2E};$$

which will be the expression for the total elastic reaction of the bent column. Dividing by $\frac{d^2\pi}{4}$ for the area, and substituting $r=\frac{L}{d}$ for the ratio, we obtain the elastic reaction per square inch:

$$f=\frac{9}{5r^2E}.$$

The compression per square inch any column is capable of sustaining, cannot, theoretically, exceed the amount indicated in the preceding equation. Direct experiments, however, show that a somewhat higher resistance is attained in practice.

In one respect the preceding theoretical deduction is very noteworthy. The ultimate compressive resistance of the material, it appears, does not enter into the question of the resistance of a long column at all. The strength of the column will be directly proportional to its stiffness, no matter whether the material be the softest wood, or the hardest steel. This theoretical conclusion is most satisfactorily corroborated by the results of experiment. The stiffness of wrought iron is, in round numbers, double that of cast iron, and 30 times that of wood, and by reference to the final tables of the strengths of the columns, at the ratio of 80, it will be seen that the resistances of 81 cwt. per square inch for wrought iron, 40 cwt. for cast iron, and 2½ cwt. for wood, hold very nearly the same relation to each other as the stiffness of the respective materials. Below a certain ratio, however, the ultimate resistance of the material will, to a greater or less extent, govern that of column. If C represent the crushing strength of the material per square inch; and f as before the elastic reaction, we have the breaking weight per square inch,

$$\text{B.W.} = \frac{Cf}{C+f}.$$

Substituting the value of,

$$f = \frac{9}{5r^2E}$$

in the preceding equation, the formula for the theoretical resistance per square inch of a round column of any length or diameter becomes:

$$\text{B.W.} = \frac{C}{1+\dfrac{5r^2CE}{9}}.$$

For cast iron C = 41 tons per square inch, and E = .00018, for wrought iron C = 17 tons, and E = .0001, and for *steel* $C = 30$ to 51 tons, and E = .00008 to .000075;

hence the value of the fraction $\frac{5CE}{9}$ will be $\frac{1}{245}$ for the former metal, $\frac{1}{1060}$ for wrought iron, and from $\frac{1}{730}$ to $\frac{1}{480}$ for steel. In the instance of the two first metals we already know the value of the fractions, as derived from direct experiment, to be $\frac{1}{330}$ and $\frac{1}{2000}$ respectively, and we may reasonably anticipate an equivalent reduction of the theoretical fraction will be obtained in the instance of steel columns. On the basis of the experiments on wrought-iron columns, the specific value of the fraction for mild steel would, since the strength is about $1\frac{3}{4}$ times and the stiffness $1\frac{1}{4}$ times that of the former metal, be:

$$\frac{1\times1\frac{3}{4}}{2000\times1\frac{1}{4}}=\frac{1}{1400};$$

and, in the same manner, for strong steel it would be:

$$\frac{1\times3}{2000\times1\frac{1}{3}}=\frac{1}{900}.$$

The formulæ for the ultimate resistance of steel columns in tons per square inch would, therefore, be as follows:

$$\text{Mild steel.}\quad \text{B.W.}=\frac{30}{1+\frac{r^2}{1400}}.$$

$$\text{Strong steel.}\quad \text{B.W.}=\frac{51}{1+\frac{r^2}{900}}.$$

Considering the identity of many of the attributes of wrought iron and steel, and the close relationship existing between the two metals generally, it will be unnecessary to test the preceding formulæ by more than a limited number of well-considered experiments. The ultimate resistance of steel to compression has already been ascertained to as great a degree of exactness as *can*, from the nature of the question, possibly be attained; it only remains, therefore, to define the limits of the series in the other direction, that is to say, the strength of a column of such a length that failure takes place entirely by bending.

To obtain the necessary data for so doing, the author instituted a series of experiments on flat steel bars, of thicknesses ranging from $\frac{1}{100}$ to $\frac{1}{400}$ of the length. In all instances the ends of the bars were slightly rounded, in order to give greater play to the elasticity of the material; the resistance per square inch of the bars consequently represented the breaking weights of *square* columns, unfixed at the ends, at ratios increasing from 100 to 400. We have already shown that the maximum bending moments on such columns are, theoretically, three times greater than they would be if the columns were fixed at each end; and Mr. Hodgkinson's experiments show that the same conclusion holds good in practice. Hence, to make our formulæ for round steel columns fixed at the ends applicable to the bars submitted to actual test in the present instance, it will be necessary to multiply the fraction representing the bending element in the formula by three, in respect of the columns being unfixed at the ends; whilst, on the other hand, as the strength and stiffness per square inch of a square beam is $\frac{1}{3}$rd greater than that of a round one of equal depth, it will be necessary to increase the fraction *in that* ratio, on account of the modified form of cross section. The formulæ thus adapted to square columns unfixed at the end will be as follows:

$$\text{Mild steel.} \quad \text{B.W.} = \frac{30}{1+\frac{r^2}{620}}.$$

$$\text{Strong steel.} \quad \text{B.W.} = \frac{51}{1+\frac{r^2}{400}}.$$

In carrying out the experiments on these steel bars, the first step taken was to ascertain the value of the modulus of elasticity E, in each instance. With the thinner bars this was obtained by noting the mean deflection induced by the weight of the specimen *per se*, and substituting that value of δ in the equation,

$$E = \frac{51200d^2\delta}{S^4}.$$

For ratios below 200, the deflection under a weight W applied at the centre was taken as the measurement of the elasticity; consequently in those instances,

$$E = \frac{4d^3b\delta}{S^3W}.$$

The value of E so deduced was found to range from .000077 to .00006. The exact value for each bar was first recorded, and then the specimens were successively submitted to direct test, by the application of weights taking effect immediately on their ends, without the intervention of levers of any sort. The resistances per square inch exhibited by the different bars were so unexpectedly consistent throughout, and the respective amounts are so closely represented by the formulæ already advanced, that no modification of the latter is suggested by these experiments. Selecting two extreme cases to illustrate the maximum extent of divergence between the calculated and experimental results, we have for the comparatively weakest column, one in which the ascertained value of E was .00007, and that of $r=266$. Under a strain of 4.8 cwt. per square inch this column bent laterally some 9 inches, and would have fractured under the same stress had not the further descent of the weight been purposely arrested. Supporting the bar on the side which had bulged out, by strutting it in such a manner that it was free to bend in the reverse direction only, the ultimate resistance proved to be $5\frac{1}{3}$ cwt. per square inch; so that in one direction the steel was evidently 10 per cent. stronger than it was in another. Now, by the formula the resistance should have been,

$$\text{B.W.}=\frac{51}{1+\frac{266^2}{400}}=.286 \text{ tons}=5.7 \text{ cwt.}$$

per square inch, which is $\frac{9}{10}$ths cwt. greater than that derived from actual experiment.

Proceeding in the same manner with the strongest column tested, in which the value of E was .00006, we have $r = 364$, and the ultimate observed resistance per square inch $= 3\frac{1}{4}$ cwt. By the formula the resistance should have been:

$$\text{B.W.}=\frac{51}{1+\frac{364^2}{400}}=3.1 \text{ cwt.}$$

per square inch, or about $\frac{1}{7}$th cwt. less than that actually attained.

In no instance did the percentage of variation between the calculated and experimental ultimate resistances of the steel bars, exceed the limits indicated by the two preceding experiments; it is, therefore, unnecessary to extend our analysis of the results.

The behaviour of these thin bars, none of which exceeded $\frac{1}{2}$ in. in thickness, under the loads imposed, affords most conclusive evidence of the vast superiority of steel over every other known material, wherever severe strains, of any nature, have to be encountered. With some 95 per cent. of the ultimate load piled upon a column, a very slight lateral pressure sufficed to deflect it many inches, and the appearance of insecurity was uncomfortable to a degree. Removing the abnormal stress, however, the bar would instantly recover itself, and after a few counter vibrations finally come to rest in its original vertical position. Under stresses higher than the preceding, the load was sustained by the column indifferently whether it maintained its original form, or was bent one inch, or even six inches; the results of the experiments in this particular corroborating, in a very interesting manner, the theoretical deduction that the elastic reaction of the bar would be a constant quantity for all degrees of deflection.

It is, therefore, as we have already observed, unnecessary to alter the provisional formulæ for the strength of steel columns, which was derived from experiments on wrought-iron bars and tubes. It will be well, however, to give them in the same form as the formulæ for iron and wood columns. We have, therefore, for the breaking weight in cwts. per square inch if round columns:

$$\text{Mild steel.}\quad \text{B.W.} = \frac{840{,}000}{1400 + r^2}$$

$$\text{Strong steel.}\ \text{B.W.} = \frac{920{,}000}{900 + r^2}$$

The results given by these formulæ for the several ratios enumerated below are as follows:

Ratio	5	10	15	20	25	30	35	40	50	60	70	80
Mild steel. B.W. cwt....	590	560	515	465	415	365	320	280	215	168	133	107
Strong steel. B.W. cwt.	995	920	820	710	602	510	432	368	270	204	158	126

We may be allowed to refer here briefly to one experiment, purposely excluded from the preceding series, because, although perhaps of little practical importance, it will at least serve to mark the limits of resistance which a column of any known material can with reason ever be expected to attain. At the same time it will afford additional evidence of the truth of the hypothesis upon which the rules for the strength of columns, given in this investigation, are based.

A blade of fine steel $\frac{1}{16}$th of an inch only in thickness, and 1300 times that dimension in length, hardened and tempered to a brown shade, was fixed securely at each end in the testing apparatus, in such a manner that no direct flexure could possibly take place, without the due proportions of contrary flexure being also induced. Under these conditions the ultimate thrust sustained by this thin blade of steel, proved to be upwards of 1 cwt. per square inch, an unprecedentedly high amount, when we consider that the ratio of length to thickness in this instance was the enormous one of 1300 to 1. The value of E was found by experiment to be .000052; hence the stiffness of the steel was about $1\frac{1}{2}$ times that adopted in our formula for strong steel columns. By that formula the ultimate resistance of a *round* column of the given ratio would have been

$$B.W. = \frac{920000}{900 + 1300^2} = .54 \text{ cwt.}$$

per square inch. For a square column, as we have already shown, it would have been $\frac{1}{3}$rd greater $= .72$ cwt. per square inch. But the stiffness of the steel tested was one half greater than usual, and it has been demonstrated that the strength of a long column is exactly proportional to the stiffness of the material; hence the ultimate resistance indicated by the formula would have been $.72 \text{ cwt.} \times 1\frac{1}{2} = 1\frac{1}{10}$ cwt. per square inch about, which was the identical amount obtained in the experiment.

It is necessary now to consider briefly the extent to which the resistance of a column will be governed by the form of its cross section.

Section 22.

ON COLUMNS OF VARIOUS FORMS OF CROSS SECTION.

The most convenient method of effecting the necessary modifications in the formulæ already advanced to render them applicable to columns of any given form of cross section, will be to substitute for the actual ratio a ratio of equivalent strength, assuming the standard cross section to be circular for iron, and square for wood; because, in that case, the breaking weight may be taken direct from the tables, or, better still, according to the writer's experience, from an enlargement of the diagrams. The cross sections likely to be met with in practice are—squares, rectangles, and circles, both solid and hollow, and varieties of the forms commonly known as tee, angle, channel, cruciform, and girder irons. We shall now estimate the comparative value of these different forms of cross section, taking the strength of a solid round column as the unit of measurement.

Now the lateral strength of a square cross section is not uniform; it varies between the limits of 1 and $\sqrt[2]{\frac{1}{2}}$, and is least when the force is applied in the direction of the diagonal, and greatest when applied parallel to one of the sides. But the ultimate deflection also varies between the limits of 1 and $\sqrt[2]{\frac{1}{2}}$, and is least in the direction of the diagonal; consequently, as the strength of a long column varies directly as the stiffness, that is, directly as the lateral strength and inversely as the deflection, a square column will be of uniform strength, and be equally liable to bend in either direction. The weakness of the unsupported material at the arris does, undoubtedly, give a bias to failure in that direction, a fact evidenced in most of the experiments on wooden columns. Of course, if the cross section be rectangular, with the breadth greater than the thickness, bending would take place in one direction only; otherwise the conditions are the same as those obtained in a square cross section with sides equal to the least dimension of the rectangle. Now both the strength and stiffness of a square cross section is to that of the enclosed circle as 1.7 : 1; consequently, the strength per unit of area will be as $(1.7 \times .7854) : 1$, or as 4 : 3. The ratio of equal strength per square inch for square and round columns, respectively, will therefore be as $\sqrt[2]{4} : \sqrt[2]{3}$; that is, as 1 : .866. So, if we wish to ascertain the strength per square inch of a square iron or of a round wooden column, we have only, in the first case, to multiply, and, in the second, to divide, the actual ratio by .866, or $\frac{7}{8}$ nearly, and we obtain the ratio of equivalent strength, the corresponding breaking weight of which may be read off at once with a scale from the diagram. The strength of a hollow square cross section is, *theoretically*, stronger than a hollow circular one in the same proportion as the *solid* square exceeds the *solid* circular; by experiment it appears to be weaker; we shall, however, in this instance take the same strength per square inch for both, assuming that proper precautions will be taken to guard against buckling, by the insertion of stiffeners and gussets, if the metal be thin in proportion to the diameter. Now, for a hollow round column, L being the length, D the outside and d the inside diameter, the equivalent ratio may be obtained from the equation:

$$r^2=\frac{L^2}{D^2-d^2},$$

consequently, taking the thickness (t) in terms of the outside diameter, we get

$$r=\sqrt{\frac{1}{1+(1-2t)^2}}.$$

The preceding formula serves for square and circular hollow cross sections, and the following formula will give with sufficient accuracy the equivalent ratios for the other forms of cross sections, classed as angle iron, channel iron, &c.,

$$r=\sqrt{\frac{3.\,t_1+t\,(a-t_1)}{4\,t_1+t^3\,(a-t_1)}},$$

where t_1 equals the thickness of the narrowest side of angle, tee, or cruciform iron, or the joint thickness of the flanges of channel or girder iron, t the thickness of the remaining side, or of the web of channel or girder iron, and a the ratio of the widest to the narrowest side, or of the web to the flanges. Applying this formulæ to different forms of cross sections with varying thickness of metal, we obtain the equivalent ratios given in the following tabular form, which enable us to determine the breaking weight per square inch, corresponding to either of the assumed cross sections, by the simple application of a scale to the diagram:

Solid round, 1. Solid square, .87 equivalent ratio.

Thickness of metal in terms of diameter or least side.

	.1	.15	.2	.25	
Hollow, square, or round	.78	.82	.86	.90	Equivalent ratio.
Angle, tee, and cruciform equal sides	1.19	1.17	1.15	1.13	"
Girder and channel equal sides...	1.03	1.01	.99	.97	"
Girder and channel, web double width of flange	1.19	1.17	1.15	1.13	"

The importance of obtaining a firm square bearing for the ends of compression members is, as we have already shown, well proved by theory and experiment. Occasionally in civil engineering structures, and frequently in machinery, it is impossible to comply with this condition. In such cases, should both ends be loose, the equivalent ratio would be $r=\sqrt{3}$, and if one end *only*, $r=\sqrt{2}$; the diameter, in both instances, might with advantage be reduced towards the ends; under all other circumstances the diameter should be uniform. The piston-rod and connecting-rod of a steam engine may be cited as types of the correct forms for fixed and loose-ended compression members. Even in very short lengths the advantage of a fair bearing is very marked: thus m being the maximum strain per unit of area in terms of the average, then the corresponding position of the centre of pressure (x) from the axis of a square cross section would be,

$$x=\frac{m-1}{6},$$

and for a round section,

$$x=\frac{m-1}{8},$$

so that a maximum strain of double the amount due to fair loading ($m=2$) would result from shifting the centre of pressure the fraction of the diameter (x) from the axis,

$$x=\frac{2-1}{6}=\tfrac{1}{6},$$

and if the centre of pressure should fall on the edge of the square cross section, the maximum strain would be $m=1+6x=4$ times the average.

Thus, in whatever light we view the question, evidence accumulates proving the absolute necessity of careful design, and a liberal allowance of strength, in all cases where materials are used in compression. Examples might be multiplied, but it is unnecessary, as we shall have to refer to the subject more particularly in the succeeding Section.

PART V.

ON ARCHED RIBS.

Section 23.

ON THE INDETERMINATE NATURE OF SOME OF THE STRAINS.

WHEN proceeding to the consideration of the strength of columns, we observed that the investigation would have many points in common with the preceding one on the transverse strength of beams; in the same manner the subject we are now attempting to elucidate may be said to be essentially dependent upon the whole of the data advanced and laws deduced, up to the present stage of our investigations. For an arched rib may be defined as a long column subject to a certain amount of bending stress, and its resistance will, consequently, be governed by the laws affecting the strength of beams, as well as by those relating to the strength of columns. The results of numerous experiments prove that neither in the instance of the beam nor the column do theoretical deductions strictly obtain, and we may justly conclude that the same, or some equivalent specific anomalies, will *also* be exhibited in the instance of the arched rib. It is, therefore, impossible to ascertain by any mathematical process, no matter how refined, the exact resistance of an arched rib, and since experiments directly relating to the subject have not up to the present time been carried out, any profession on our part of being able to give more than an approximate solution of the problem would merely serve to prove that we had not even taken the first progressive step, which, in this instance, as in so many others, consists in the knowledge of our own ignorance.

Under such conditions a careful consideration of similar works already executed is absolutely imperative. The successes attained and the defects evinced in many of the works of our predecessors afford evidence which it would be inexcusable to neglect. "*Scitum est periculum ex aliis facere, tibi quid ex usu siet*" is as sound a piece of advice now as it was two thousand years ago, and to no one does it more forcibly address itself than to the young engineer. The mathematical training every student of engineering now undergoes is somewhat apt to give him an excess of confidence in the exactness of his own deductions, little less objectionable in its practical effect than the timid and slavish adherence to precedent evinced by the proverbial "practical" man. He is, by the same system of training, led to form a ridiculously low estimate of the extent to which the most approved practice of the present day is indebted to inductive processes. By far the larger proportion of the varied problems the civil engineer is called upon to entertain will be most readily and satisfactorily solved by deducing the required result from certain data previously arrived at by the process of induction. In one respect the science of engineering differs from most of the other exact sciences, inasmuch as it is seldom necessary to investigate any problem extensively by induction; the practically correct, and not the theoretically exact, solution is the requirement of the engineer, and it is, therefore, merely necessary for him to trace back sufficiently far to allow of that end being attained. Thus, an analysis in a former Section, of experiments on the transverse strength of beams indicated the existence of an element of strength dependant upon the form of cross section, and varying with the nature of the material; and in the same manner, the resistance of long columns was in a subsequent Section found to be higher than theory indicated. In each of these instances it was only necessary for us to ascertain enough of the laws governing the increased resistance to enable that element to be included in the calculations, and it was not necessary, nor did we attempt, to ascertain the first cause of the anomalies exhibited in many of the experiments.

The structures falling within the province of the civil engineer are generally so far analogous as to render the ordinary plan of procedure by deduction quite justifiable; but with the mechanical engineer it is otherwise. The conditions under which his works are placed are so involved, and they are so often liable to strains of literally indeterminate amount, that any attempt to generalise would be perfectly futile. Thus, given the resistance of a cylindrical bar of wrought iron to a bending, and also to a twisting stress, who could presume from such data to deduce the proper proportions and sections of the crank shaft of an inside cylinder locomotive? It would be necessary to eliminate the bending stress, and that alone would be made up of the several elements due to the reaction of the driving wheel springs on the roughest portions of the line, of lateral blows on the flanges of the wheels, of the steam pressure on the piston, and of the *vis viva* of the gear and the crank shaft itself. When we consider the *form* of the member which has to sustain these constantly varying stresses, in conjunction with torsional and other strains, it is very apparent that the necessary section and the most economic distribution of the material could only be arrived at by a strictly inductive process. Who, again, could attempt to compute the section of tyre desirable for the driving wheels of the same engine? and yet the practical identity of the proportions adopted by the different makers in this and in other countries, both for the tyre and the crank shaft, clearly prove that these apparently indeterminate problems have been solved by induction, with as much exactness as the tensional strength of a two-inch square bar may be arrived at by deduction from that of an inch square bar. These considerations should prove reassuring to the civil engineer when confronted with problems of more than average intricacy, such as those relating to the subject of the present Section.

Complex formulæ are rarely resorted to by experienced practical men, the reason for which will probably be found in the fact that too great attention to minutia in the calculations is apt to make a man imagine that the chief portion of his work is accomplished when he has solved the problems relating to strains in the structure, and so induce him to overlook some apparently trifling practical detail, the neglect of which might, after all, be of sufficient moment to cause his work to be classified as a warning rather than as an example. No intelligent engineer, when designing a complicated structure, hesitates to throw over entirely his theoretical deductions, should the proportions thus determined appear unsatisfactory to the eye, because experience has taught him that, in nine cases out of ten, the discrepancy between the proportions deduced and those anticipated is due to the falseness of the hypothesis upon which the theoretical conclusions were based, and not to any error of judgment in that very trustworthy guide—the experienced eye.

We doubt whether that historical *faux pas*, the continuous fish-bellied rail, is justly credited to a member of our profession. We think the idea must have originated with some worthy mathematician, who fondly imagined that perfection was approached in exact proportion to the number of decimal places in his calculated ordinates. A purely practical man would, we feel sure, have instinctively rejected the pseudo-scientific form, although he may not have been able to point out the error in the hypothesis to which the vicious result was due.

Many other instances might be cited where purely theoretical deductions are equally at fault: as a case in point we may refer briefly to the bracing of a bow-string bridge. Numerous and elaborate formulæ are to be found in French and German text books, professing to define the exact value of the maximum strain occurring upon each tie and strut under the passage of a moving load, but unfortunately in nearly every instance these formulæ are based upon the hypothesis that the system is perfectly rigid, or, in other words, that the length of each member of the bridge will remain constant under every degree of strain. Now, we know that some deflection must necessarily follow the application of each successive increment of load, however small; and we know also that such deflection could not take place unless the length of every member in the bridge differed, plus or minus, from its original amount. This, of course, is due to the elasticity of the material, and the complicating conditions thus introduced are so embarrassing, varying as they do with the relative sectional areas of arched rib, tie, and bracing at each portion of the length, that it would be an almost endless task to deduce the exact mathematical value of the maximum strain occurring upon each member of a bowstring bridge under a rolling load of known intensity. No intelligent engineer wastes his time in the investigation of such problems, because he knows that after all the result of his labour would be of no practical value.

It would have been assumed that the whole of the metal in the bridge was in a state of molecular equilibrium in the absence of any external stress, or else that certain members were subjected to initial strains of known intensity. To enable the first hypothesis to obtain, the bars and plates would all have to be of precisely the required length, and accurately fitting; whereas, in practice, we know that in the process of riveting one bar would be drawn up tight, whilst, probably, the adjacent one would be quite slack, or even present a permanent lateral curvature. Or, again, in putting the girder together it may have been necessary to "draw out" one of the ties to make it fit into its place, and, probably enough, it would have been riveted up when quite hot; but this apparently trifling, and certainly unavoidable, practical contingency would be sufficient in many instances to augment the laboriously deduced theoretical strain as much as 50 per cent. If cotters were introduced, either for the purpose of securing uniform or initial strain, the value of the result would be entirely dependant upon the intelligence of the men entrusted with the cottering up; and with the best arrangements the degree of uncertainty would be little less than the previous instance. It is quite possible to obtain, by mere inspection, the strains on the several members of a bowstring bridge within 25 per cent. of the amount which would be given by the most elaborate mathematical analysis; consequently, remembering that it is impossible to make sure in practice of approximating even within 50 per cent. to the exactly deduced result, we think the adoption of an elaborate method of calculation for the minor members of a bowstring bridge, and for similar reasons for the strains upon an arched rib, only serves to prove that the conditions of the problems are imperfectly appreciated by those attempting to solve them.

That the metallic arch presents so many more difficulties to the investigator than its prototype the stone arch, is due entirely to the comparative elasticities of the two materials. In the former instance, the strain upon the structure is most sensitively dependant upon the laws of elasticity, whilst in the instance of the stone bridge the same laws are practically inoperative. If the latter conclusion held good with metallic structures also, many engineering problems would admit of an infinite number of solutions. For example, in a continuous girder, the sectional area over the support might be any fraction, or any multiple, of that at the centre, provided that the sum of the two areas was always a constant quantity. Again, by reason of the elasticity of the material, the *ultimate* resistances and the *elastic* resistances of different structures may give very antagonistic ideas of the relative strengths of the same. Thus, if the strain on the material under a given load be 5 tons per square inch, and the ascertained ultimate resistance of the same be 25 tons per square inch, it might, at first sight, be supposed that the breaking weight would be simply five times the load inducing the strain of 5 tons per square inch. Upon more mature consideration, however, it will be seen that the apparently convertible terms, working load$=\frac{1}{5}$ breaking weight, and working strain$=\frac{1}{5}$ ultimate resistance, are in reality anything but synonymous expressions.

We may illustrate the preceding proposition by a simple example: Given a wrought-iron bar, firmly suspended in a vertical position by its upper end to a rigid beam, and accurately butted at its lower end against an incompressible bed-plate, then upon the application of an annular weight to the centre of the bar a tensile strain would occur upon the upper half of the bar, and a compressive strain upon the lower half; and, since the extension and depression of wrought iron is practically the same under similar unit strains, the tension and compression upon the bar would, under the assumed conditions, be of equal intensity. Assuming the bar to be too short to admit of flexure in the lower half, its *elastic*, and also its *ultimate*, resistance would be about double the amount due to the sectional area. Now, let us suppose that an error has been made in the adjustment of the bar, so that it does not butt against the bedplate by a distance equal to $\frac{1}{2000}$ of the length; then it would follow that the full elastic resistance of the upper half of the bar would be attained before any strain occurred upon the lower half. The effective *working* strength of the bar would, therefore, by reason of the error in its adjustment, be one-half only of the former amount. Proceeding with the loading, however, beyond the elastic limit, the lower half of the bar would gradually come into play, and in consequence of the relatively rapid rate of extension of wrought iron beyond the elastic limit, the lower half of the bar would be strained up to that point before the upper half had received even 10 per cent. additional pull; and long before the ultimate resistance was attained the relative strain upon the upper and lower halves of the bar would be dependant upon the quality of the metal alone, irrespective of any practical error in the first adjustment.

It appears, then, that whilst the *elastic* resistance of the accurately adjusted bar would be about *one-half* of its ultimate resistance, that of the bar in which an error amounting to $\frac{1}{2000}$ of the length had been made, would be *one-fourth* only of the same amount. We have taken the simplest example for the purpose of illustration; but there are many more complicated cases to be met with in the ordinary practice of an engineer where the same conclusions must necessarily obtain, although the process of reasoning may not be equally apparent.

More especially, when dealing with arched ribs, are these considerations of the utmost moment to the engineer. In many respects an arched rib is analogous to a long column; and it is, therefore, necessary to take the *ultimate* resistance of the bridge as the measure of its strength, and not the *working* strain per square inch on the metal, considered in relation to the ultimate resistance per square inch. For, although a weight equivalent to an *average* compression of 5 tons per square inch may be sustained by a column of certain length without straining any fibre beyond the limits of elasticity, it does not necessarily follow that an additional ton per square inch would not suffice to effect the destruction of the column. It would, consequently, be very impolitic to consider the specific amount of the mean compression per square inch as any guide to the desirable proportions of such a column.

We have said that the same conclusion applies to arched ribs, but this truth is not universally admitted; in fact, Colonels Eads and Flad, and Mr. Pffeifer, engineers of the St. Louis, 550 ft. span, steel arched bridge, contested strongly that expression of our views on a former occasion. Economy, however, was a great *desideratum* in the St. Louis bridge, so it is perhaps only natural that the engineers should have adopted the most favourable hypothesis. That American engineers have no insuperable objection to deriving their factor of safety from the breaking load of the *structure* itself, instead of from that of the *metal* of which it is composed, is evidenced by the example of the Ohio Suspension bridge. Although, as maintained by Mr. Roebling, the cables of that bridge, whether straight or curved, must take up their fair proportion of the work before *failure* could take place, it does not therefore follow that the *working* strain would be shared by them in similar proportions. In this instance the hypothesis adopted was diametrically opposed to the one assumed by the engineers of the St. Louis bridge. In one respect, however, the identity was perfect—it was the most favourable to the structure under consideration.

The evidence advanced in the present section has been selected with the view of establishing our proposition that infinitesimal calculations, however interesting to the mathematician, are of little use in practical engineering, since mathematical accuracy avails us nought when the truth of an hypothesis is always questionable, and in no case admits of demonstration.

Section 24.

ON THE PROPERTIES OF THE CURVE OF EQUILIBRIUM.

The first step necessary to be taken when computing the strains occurring on different parts of a bridge, under a given distribution and intensity of load, is to ascertain the value of the bending moments at the several points. A little consideration will make it evident that this preliminary stage of the investigation will be the same for all types of bridges; that is to say, we may proceed with the calculations so far before even the class of bridge, whether girder, suspension, or arch, has been determined. For, the moment of a weight is simply the product of that weight into the effective leverage at which it acts, and, as every schoolboy knows, the power of a lever is governed solely by the relative lengths of the two arms measured in straight lines from the points where the forces are applied to the fulcrum. Although it is apparently a self-evident proposition, that the power of a lever will be unaffected by its actual *form*, some curious blunders are sometimes perpetrated in that respect. We were once consulted by an ingenious fitter, who had been "going in" for mechanics, about an improved form of lever which he had devised, in which compactness was proposed to be attained without loss of power by twisting the long arm spirally, so that it presented the appearance of a Brobdignagian corkscrew. Doubtless it was by a similar process of reasoning that an equally ingenious patentee conceived the brilliant idea of doubling the power of our steam engines by corrugating the surfaces of the piston, in order to expose a greater area for the steam pressure to operate upon. Equally absurd, though less grossly apparent, were many of the fallacies advanced in the numerous pseudo-scientific investigations of the loss of power involved in the use of the crank, published at the period when rotary engines were engaging the attention of nearly every mechanical engineer. Mistakes such as those we have cited would not, of course, be perpetrated at the present time even by an ordinary student of engineering; but still when dealing with very complicated systems of trussing, even the most experienced engineers are apt to be embarrassed, unless the fundamental principle be kept well in view, that the direction and the amount of the force taking effect at any given point is the same as it would be were the arrangement of the parts the simplest conceivable.

Our first duty, therefore, is to devise some convenient method of arriving at the values of the bending moments occurring at different points of a bridge, under a given distribution and intensity of load. The relative values of the bending moments will be rendered the most apparent, and the results will be the best placed for reference, by exhibiting them on a diagram. This being so, although there are very many different ways of arriving at the required values of the bending moments, the most convenient and the most direct will be the one in which a single process only is necessary, such as the construction of a diagram from which the value of the bending moment at any point may be obtained by simply scaling.

The method of construction, found by ourselves the most generally advantageous in the ordinary routine of practice, will be best exhibited by a specific example; we propose, therefore, to adopt that process.

Given a bridge 100 ft. span, loaded with 2 tons per foot run over one half of the span, and with 1 ton per foot over the other half, required to exhibit on a diagram the value of the bending moments at all points.

Now the distributed load may be replaced by equivalent concentrated loads at any convenient number of points, without affecting the results of the calculation; in most instances it will be sufficient if 10 equidistant points on the span are taken, since, with that number of ordinates the curve of moments may be readily described. Proceeding in this manner with the example selected, the concentrated weights at 10 ft. intervals equivalent to the distributed load of 2 tons per foot run, will of course be $10 \times 2 = 20$ tons; whilst for the lighter half of the bridge they will be 10 tons, and for the centre weight the mean between the two, or 15 tons. The next stage in the process is to compute the moment of each weight round the point of support to which it is transmitted; and since, from the principles of the lever, it is essential to the equilibrium of the system that the sum of the moments of the several weights transmitted to each support should be equal, to obtain the maximum bending moment exerted by the load, it will only be necessary to proceed simultaneously with the weights on each side of the centre, commencing with those nearest the points of supports, and to carry on the process until the last weight has been dealt with. The probability is that, in order to comply with the conditions of equilibrium, or to obtain equality of the moments, this last weight will have to be subdivided into two portions, which we will term x and y, to be transmitted to the heavily-loaded, and to the lightly-loaded supports respectively. The process of computation so far is exhibited below:

HEAVILY-LOADED SIDE.

Weight.	Distance from support.	Moment.	Sum of moments.
20 tons	× 10 feet =	200	200
20 "	× 20 " =	400	600
20 "	× 30 " =	600	1200
x "	× 40 " =		

LIGHTLY-LOADED SIDE.

Weight.	Distance from support.	Moment.	Sum of moments.
10 tons	× 10 feet =	100	100
10 "	× 20 " =	200	300
10 "	× 30 " =	300	600
10 "	× 40 " =	400	1000
15 "	× 50 " =	750	1750
y "	× 60 " =		

It will be seen that the sum of the moments of the first three heavy weights on the left hand side is 1200, and of the first five light weights, 1750. Now there only remains one other weight of 20 tons to be dealt with, and we know that in order to comply with the conditions of equilibrium, the values of x and y, into which it is subdivided, must be such that the ultimate sum of the moments on each side may be equal. To ascertain the value of one of these portions, say of x, we may proceed as follows :—To make the sum of the moments on the heavy side equal to the present value of the same element on the light side, it is evident that a proportion of the 20 ton weight equal to,

$$\frac{1750-1200}{40}=13.75 \text{ tons,}$$

must be appropriated to the heavy side alone, and this leaves us only a weight of $20-13.75=6.25$ tons, which will be transmitted to each support in proportions varying inversely as the distance of the weight from the support. Hence, the proportion included in the value of x will be:

$$\frac{6.25 \text{ tons} \times 60 \text{ feet}}{100 \text{ feet}}=3.75 \text{ tons.}$$

We have, therefore, $x=13.75+3.75=17.5$ tons, and it follows that $y=20-17.5=2.5$ tons.

Substituting these values of x and y in the former table, we obtain the complete series of moments as exhibited on the opposite page.

HEAVILY-LOADED SIDE.

Weight.	Distance from support.	Moment.	Sum of moments.
20 tons	× 10 feet =	200	200
20 „	× 20 „ =	400	600
20 „	× 30 „ =	600	1200
17.5 „	× 40 „ =	700	1900

LIGHTLY-LOADED SIDE.

Weight.	Distance from support.	Moment.	Sum of moments.
10 tons	× 10 feet =	100	100
10 „	× 20 „ =	200	300
10 „	× 30 „ =	300	600
10 „	× 40 „ =	400	1000
15 „	× 50 „ =	750	1750
2.5 „	× 60 „ =	150	1900

It appears, then, that the maximum bending moment exerted by the load in the instance selected for illustration is 1900 foot tons; and to obtain the moment at any other point of the span, no further computation is necessary, since all the required elements for the construction of a diagram exhibiting the relative values are contained in the columns headed "Sum of Moments" in the preceding Table.

The method of construction will be as follows: Set off (Fig. 31,) on the span A—B, the points *a*, *b*, *c*, . . . at which the distributed load has been assumed to be concentrated, and square up the lines *aa'*, *bb'*, *cc'* . . . and also the lines A C and A D. On the line A C set off to any convenient scale the "Sum of Moments" on heavily-loaded side, measuring in all instances from the point A, so that A f=200, A g = 600, and A h=1200; and in the same manner set off on the line

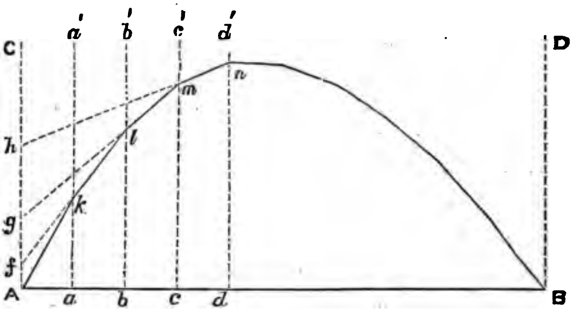


FIG. 31.

B, D, the "Sum of Moments" on the lightly-loaded side. On the line *dd'*, where from the Table it is seen the maximum bending moment occurs, measure off a height *d n*, representing that moment (1900) to the same scale as before. Then from the point *n* draw the line *h n* intersecting *cc'* at *m*, and in the same manner from the point *m* draw the line *gm* intersecting *bb'* at *l*, and from *l* the line *f l* intersecting *aa'* at *k*, and from *k* the line A*k*; then A*k l m n* will be the curve of moments for the heavily loaded side, and in the same manner that for the other portion of the span may be constructed. To obtain the bending moment in foot tons occurring at any point on the span, it will only be necessary to scale the ordinate of the curve of moments at the given point.

The curve thus arrived at, it may be readily shown, is made up of portions of two parabolas, one for the heavily-loaded side, and the other for the lightly-loaded side. The former will be a portion of the parabola which would represent the bending moments if the heavy load were distributed over the entire length of the spans, and the latter a portion of the parabola representing the moments if the light load were so distributed. Now, when the height of a parabola is proportionally small, say about $\frac{1}{10}$th of the chord, the divergence between that curve and a segment of a circle of the same ver. sin. and chord is very small, and, as far as the present case is concerned, quite inappreciable. In practice it is sometimes convenient to take advantage of this identity of the two curves, and so save the labour, small though it is, of describing the mathematically exact one. Thus, let it be required to represent on a diagram the bending moments occurring on a span of 100 ft., subjected to an uniformly distributed load of 1 ton per foot, and to a rolling load also of 1 ton per foot, the latter load advancing from one end of the span by successive distances equal to $\frac{1}{10}$th of the span. To exhibit the required series of bending moments eleven curves will have to be described; one when no rolling load is upon the bridge, a second when $\frac{1}{10}$th of the span is covered, a third when $\frac{2}{10}$ths is covered, and so on until the entire bridge is covered by the rolling load. By the following process the required bending moments at every point of the span, under the eleven different distributions of the load, may be obtained in less time than it takes to read the description.

The maximum bending moment will occur at the centre of the span when the bridge is entirely covered by the rolling load, and it will, since the span is 100 ft., and the total load 2 tons per foot, be equal to:

$$\frac{100^2 \times 2}{8} = 2500 \text{ foot tons.}$$

No further calculation is required. The construction of the diagram will be as follows: At the centre of a base line representing the span raise a perpendicular, and measure off on that line the maximum bending moment of 2500, to such a scale that the height may be equal to $\frac{1}{10}$th of the span, or thereabouts. Select an ordinary cardboard or wooden curve, by which a circle may be described through the three given points, namely, the two extremities of the span and the height set off at the centre; then the ordinates of that curve will represent the bending moments at all points of the span under the same conditions, and to the same scale as the computed moment of 2500 foot tons. Now, when the rolling load is entirely off the bridge one-half of the weight will have been removed, and consequently the maximum bending moment under those conditions will be one-half of the former amount, and the ver. sin. of the curve of moments will obviously be also the same fraction of its former value. But within certain limits the ver. sines of curves with a common chord vary inversely as the radii; hence the radius of the curve representing the bending moments when the span is lightly loaded, will be double that of the one used to describe the former curve of moments.

The two curves thus selected are the only apparatus required to enable us to construct the curves of moments under all the given distributions of the load. For, as we have before observed, the curve corresponding to the portion of the span heavily loaded will always be of the same radius, no matter how far the rolling load extends on to the bridge; and in the same manner the curve for the lightly-loaded portion will always be, in the instance selected for illustration, double that radius. The change of radius, consequently, must take place at the same point as the change in the intensity of the load; hence, in order to describe the curve of moments when the span is partially loaded, say one-half covered by the rolling load, it is merely necessary to take the two curves (Fig. 32), rest them against a couple of pins fixed at the extreme ends of the span, and raise them up till they have a common tangent at the centre of the span, when the lightly-loaded portion may be drawn in by the aid of the curve of larger radius, and the remaining portion by that of the other curve. In the same manner, to describe the moments when the rolling load covered $\frac{1}{10}$th of the span, the curves would have to be lowered till they had a common tangent at the same point, and so on for any other distribution of the load. The construction of the whole series of diagrams would, therefore, be the work of a few minutes only.

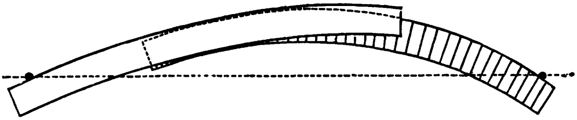


FIG. 32.

A neater and more precise way of describing the diagrams may be obtained by substituting for the two separate cardboard curves a single curve, one-half of which is struck to the larger and one-half to the smaller radius. By this means the constant shifting of the curves required to comply with the condition of their having a common tangent at the point where the intensity of the load changes is obviated.

It may appear at first sight, that the determination of the bending moments under various distributions of the load must be a work of supererogation, since the maximum bending moments for all points occur when the rolling load entirely covers the span. In some respects this conclusion holds good: for instance, if the purpose of the diagram be, as it perhaps most frequently is, to enable us to proportion correctly the sectional areas at different points of the flanges of an ordinary girder, no more than one diagram would be necessary, since all others must necessarily be included within the one showing the bending moments under the maximum intensity of load. But if it be also necessary to deduce from the diagrams the maximum strain to which any portion of the "web" will be subjected, one diagram will not suffice, and, for similar reasons, it would not enable us to arrive at the strains occurring upon an arched rib under a rolling load, the point in which we are at present more immediately interested.

The work proposed to be effected in the present section was to provide some convenient method of arriving at the bending moments occurring on different points of a span under any given distribution of the load; and a simple system by which the required information may be ascertained and exhibited on a diagram by one and the same process has now been illustrated. Before we make an application of the diagrams to the determination of the strains upon arched ribs, it will be well, by a few illustrations, to justify our former statement, that up to the present stage the process of calculation would be the same for all classes of bridges.

The information supplied by the diagram is the amount in foot tons of the bending moments at all points of the span, and to comply with the conditions of equilibrium we know that the structure must offer an equivalent resistance; hence, for any girder the diagram represents in the same terms the product of the moments of resistance of the cross section into the unit strain. Thus, for an ordinary flanged girder the horizontal strain upon the flange at any point in tons will be the amount exhibited upon the diagram divided by the depth in feet of the girder at the same point; consequently, if the girder is of uniform depth the sectional area of the flanges will vary as the ordinates of the curve of moments; and for equally obvious reasons, if the depth varies as those ordinates, the horizontal strain upon the flanges will be constant throughout.

But uniform horizontal strain is the condition of the curve of equilibrium, hence the curve of moments constructed by the method advanced exhibits also the curve of equilibrium. That is to say, if we were to secure a flexible string at each end after the manner of a suspension bridge chain, and attach weights similarly distributed to those taken in the case selected for illustrating the construction of the diagram, the form assumed by the string, if of the proper length, would be identical with the curve of moments exhibited in Fig. 31, although it would, of course, occupy an inverted position. It follows that the curve, as represented in Fig. 31, gives us also the outline of an equilibrated arch, and in many of the earlier text-books on arches in masonry it was advised that the curve should be ascertained by the method of suspending weights to a flexible string in the manner referred to; but, of course, when a diagram can be constructed as readily as it may by the process advanced in this section, the adoption of such a method would involve an unjustifiable waste of time. The horizontal pull of the string, or thrust of the arch, would be equal to the maximum bending moment divided by the deflection of the one or the rise of the other; and the direction and relative intensity of the force at any point of the curve could, of course, be obtained by drawing a tangent to the curve at the same point.

Before we further illustrate this branch of the question, however, we will give one example of the application of the diagram in the instance of a simple girder, in order to show that all necessary information for the determination of the strains is contained therein. For this purpose we will refer once more to the 100 ft. span, loaded with 2 tons per foot run over one half, and with 1 ton per foot over the other; and we will assume the depth to be 10 ft., and the trussing as shown in Fig. 33. For convenience of reference the curve of moments (Fig. 31) is repeated on Fig. 33 to the same vertical and horizontal scale, and by the following method the strain upon all the members of

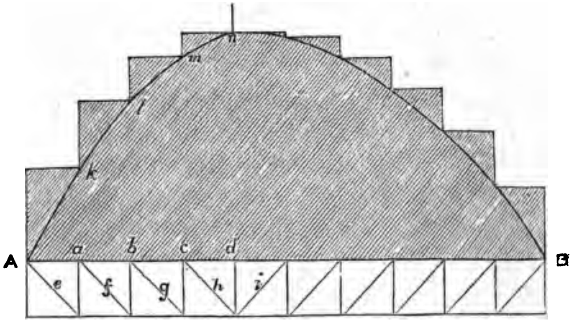


FIG. 33.

the girder may be obtained from the diagram by simply scaling.

The depth of the girder being 10 ft. throughout, the strain upon the flange at any vertical will be represented by the ordinate of the curve at the same vertical, to a scale ten times greater than that used in plotting the bending moments. Now, the curve of moments assumes the web to be continuous throughout, whilst, in this instance, it is divided into ten bays, the strain in which must obviously remain constant between any pair of verticals, since there is no agency at work between those points capable of increasing or diminishing the force; hence, the first step to be taken will be to modify the curve of moments in accordance with this condition. Thus, the strain at *a* will be continued to the end of the girder, and that at *b* will be constant between that point and *a*, and so on for the other bays; consequently, the strain upon the flange will be represented to scale by the shaded diagram, and, of course, the sectional area of the metal should vary in the same ratio.

We have only now to deal with the strains upon the bracing of the girder, and they are at once apparent upon the following considerations: The strain upon the flange at *a k* could only have been induced by the diagonal tie, *e*, consequently the *horizontal* strain upon that member must be of similar amount to that occurring upon the flange at *a k*; in the same manner the increase of the strain upon the flange at *b l* must necessarily be due to the horizontal pull of the tie *f*; hence it is evident that if the shaded portion of the diagram represents the strain upon the flange, the height of the successive steps in it represents the *horizontal* strain upon the diagonal ties, from which, of course, the actual strain on any one may be at once obtained, since it will hold the same ratio to the horizontal strain as the length of the bar holds to the horizontal distance included between its two ends. As the load is assumed to be applied at the top of the girder, the strain upon the struts will be less than that upon the ties in proportion to their reduced lengths. In the example selected for illustration the depth of the girder is equal to the length of one of the bays, consequently the height of any step in the diagram to the same scale as that used for the strain upon the flange gives the compression taking effect upon the strut, and the pull upon the corresponding tie will be the same amount $\times \sqrt{2}$. Ordinarily the strain upon the bracing in such an instance as the present would probably be obtained by adding together the increments of weight transmitted by the several diagonal ties to each pier, which are exhibited in the Table (page 233), showing the calculations necessary for the construction of the curve of moments. Referring to that Table, it will be seen that the *vertical* strain upon the tie *h* is 17½ tons, hence upon *g* it must be 20 + 17½ = 37½ tons; upon *f* 37½ + 20 = 57½ tons; and upon *e* 57½ + 20 = 77½ tons. The strains upon the ties thus deduced from the *vertical* components will be found the same as those derived from their *horizontal* components, exhibited on the diagram, and that is all we care to prove, since the purpose of our illustration was to show that all necessary *data* for arriving at the strains was afforded by the original diagram, a consideration of considerable importance when dealing with more than ordinarily complex structures.

Section 25.

ON THE STRAINS DUE TO THE LOAD.

The problems to be investigated in the present Section are those relating to the determination of the *probable* maximum strains occurring on different portions of an arched rib under certain given conditions. We say "probable" maximum strains, because, as we have maintained in a former Section (23), the conditions of the problems are such that no *definite* solution could possibly be arrived at by any process of mathematical investigation, no matter how refined or laborious the method adopted may be.

If we first take a comprehensive, though possibly a somewhat superficial, glance at the question now before us we shall be the better able to appreciate hereafter the relative importance of the different conditions involved. In the first place, then, it may be premised that the final strain upon an ordinary arched rib is made up of a number of increments of strain, each of which is subject to its own peculiar set of laws. Thus we have first the direct compression upon the rib as an equilibrated arch, which would theoretically be of uniform intensity per unit of area on any cross section, and the minimum value of which would be equal to the horizontal thrust of the arch. Secondly, we have the increased strain due to the action of the arched rib as a long column, and governed, therefore, by similar laws. Thirdly, the strain upon the arched rib as a girder, which will take effect when the distribution of the load is such that the curve of equilibrium, constructed by the method advanced in the last Section, does not pass through the centre of gravity of each cross section of the rib. Fourthly, the strains arising from the elasticity of the material, and the consequent change of form of the arch, and Fifthly, the strains due to variations of temperature.

As we propose dealing with the different questions involved in the broadest possible practical manner, regardless of any niceties of mathematical method, we shall investigate the general nature of the preceding strains in the following order: 1. The strains upon the arched rib due to the proportions of that member, and to the distribution and intensity of the load. 2. The strains due to change of form, including the modifications arising from any initial camber or deflection which may be given to the arched rib during erection. 3. The strains from changes of temperature; and 4, the increased strain arising from the conditions of the arched rib in some instances being analogous to those obtained in an ordinary straight unsupported column.

All the information required for the solution of the problems relating to the first branch of the investigation may be obtained from a diagram exhibiting the bending moments at different points of the span. For, as was stated in last Section, the curve of moments is identical with the curve of equilibrium, and consequently the diagram exhibits the amount and direction of the forces taking effect upon the arched rib at all points. If the curve of equilibrium coincides throughout with the axis of an arched rib of symmetrical cross section, it follows that the strain upon that member will be simply the compression corresponding to the thrust of the equilibrium curve at the same point. If the curve diverges from the axis of the arched rib, in addition to the preceding direct and uniformly distributed compression strain a bending stress must also be provided for, the moment of which will be equal to the compressive strain multiplied by the extent of divergence of the curve of equilibrium from the axis of the arch. For example, if *a b c d e* (Fig. 34) be the axis of the arched rib, and *a f c g e*

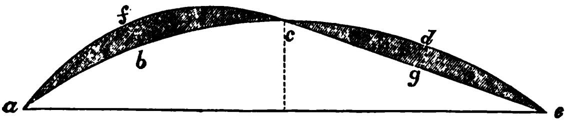


Fig. 34.

the curve of equilibrium in its correct position, the strain at *c* would be simply the compression due to the thrust of the curve of equilibrium at the same point, whilst that at *b* would be made up of two elements. Thus, there would be, in the first place, the simple compression due to the force acting at the point *f* on the equilibrium curve, and, as we have before observed, this force will be greater than the thrust of the arch, in the same proportion as the length of any tangent to the curve at the point *f* is greater than the horizontal distance included between its two ends; and, secondly, there would be the bending force, the amount of which will be equal to the product of the preceding force into the distance *f b*, measured on the normal to the curve at the point *f*. The strain per square inch on the arched rib from the simple compression would, of course, be the amount of that force divided by the sectional area in square inches; whilst that arising from the bending element would be equal to the moment of the same force divided by the moment of resistance of the cross section of rib. The latter strain would be compressive or +, on the edge of the rib nearest to the curve of equilibrium, and tensile or –, on the remote edge; and the final strains upon the rib would be the sum of the foregoing stresses. Thus, for example, let there be given an arched rib 3 ft. deep with a lattice-web, and with flanges 20 square inches in area, and, referring to Fig. 34, let the thrust in the direction of the tangent to the curve of equilibrium at the point *f* be 100 tons, and the distance $f\ b = 1.5$ ft.; then the final strain upon the metal would be deduced as follows:

The sectional area of the rib being $2 \times 20 = 40$ square inches, the uniform compression from the thrust would be:

$$\frac{100 \text{ tons}}{40} = 2\tfrac{1}{2} \text{ tons},$$

per square inch. Again, the moment of the bending force being $100 \times 1.5 = 150$ foot tons, and the moment of resistance of the cross section $= 20 \times 3'0'' = 60$; the strain per square inch from the transverse stress would be:

$$\frac{150}{60} = 2\tfrac{1}{2} \text{ tons},$$

per square inch; compression or +, on the upper flange of the rib at the point *b*, and tension or −, on the lower flange. The final strain, therefore, would be $2\frac{1}{2} + 2\frac{1}{2} = 5$ tons per square inch on the upper flange of the rib, and $2\frac{1}{2} - 2\frac{1}{2} =$ *nil* on the lower flange.

By the same method the strain on any other part of the rib might be arrived at; hence it appears that having constructed the curve of equilibrium by the simple process advanced in the last Section, and fixed its position properly with respect to the axis of the rib, the determination of the strain at any point is a mere question of scaling and multiplication. It is, therefore, only necessary now to consider upon what principles the curve is to be fixed, because it is obviously possible to construct a curve of equilibrium in an infinite variety of positions, and necessarily the corresponding strains upon the arched rib would vary to an equal extent.

Now the approximate position of the curve of equilibrium is in nearly every instance self-evident, or if it be not, may be readily made so to any one who will take the trouble to spring a thin straight edge into the form of an arch, and note the change of form resulting from the application of pressure. It would be observed that the curvature of the arch becomes flatter at the part where the pressure is applied, and sharper at the unloaded portions. Thus, if the pressure were applied at one haunch, that portion of the arch would become flatter and sink, and the opposite haunch would rise. But the curve becoming flatter is an indication that the compression upon the upper edge of the arched rib is more intense than that upon the lower edge; and from what has already been advanced in this section, it will be apparent that this again is an indication that the curve of equilibrium passes *above* the axis of the rib at that point.

Mutatis mutandis, the same reasoning applies to the portions of the arched rib which become of a sharper radius under the given distribution of the load; hence we can tell roughly where the equilibrium curve is above the axis of the rib, and where it is below it. If the *thrust* of the arch were known the position of the curve would of course be accurately defined, since it would merely be necessary to divide the maximum moment, obtained when making the calculations for the construction of the diagram, by the thrust, and the quotient would be the rise of the curve of the equilibrium in feet, and obviously the curve could then be plotted to the same scale as the arched rib, and on the same chord line. The thrust of the arch not being given, however, it is necessary to consider carefully the conditions of the problem.

For the purpose of simplicity we ignore for the present the *strains* resulting from change of form. We have, therefore, given an elastic rib confined between fixed abutments, and we will now assume that rib to be of uniform depth and sectional area, and to be pivoted at the abutments so that the centre of pressure must necessarily correspond with the axis of the rib at those points as shown on Fig. 34. It will be more convenient for our purpose to refer to a specific case; we will, therefore, take the arch loaded more heavily over the left-hand haunch, the corresponding curve of equilibrium of which is exhibited on Fig. 34. Now, referring to that figure, we know that the portion of the arched rib *a b*, where the curve of equilibrium passes above the axis, will be deflected downwards, and the portion *c d e*, where the curve passes below the axis, will be cambered upwards, under the given distribution of the load, or in ordinary engineering parlance, the heavily-loaded haunch will sink and the opposite one will rise. The horizontal distance between the two ends of the portion of the rib which is flattened out must obviously be increased; whilst, on the other hand, the remaining portion of the rib becoming cambered under the strain, the horizontal distance between its two ends will be diminished. But, since the abutments are assumed to be immoveable, one of the conditions of the problem is, that the sum of the two preceding distances be constant under all distributions of the load, and be equal to the span of the arch; hence it follows that the decrease in the horizontal distance between the two ends of the lightly-loaded segment of the arched rib, will be precisely equal to the increase in the instance of the heavily-loaded segment.

This condition enables us to fix the correct position of the curve of equilibrium as follows:

The bending strain taking effect upon different portions of the rib, has been shown to be equal to the thrust acting along the curve of equilibrium, multiplied by its distance from the axis of the rib, or, what amounts to the same thing, to the horizontal thrust of the arch multiplied by the *perpendicular* distance between the two curves; hence the total amount of bending strain upon each segment will be proportional to the areas of the spaces enclosed between the curve of equilibrium and the axis of the rib; and this rib being of uniform depth and section, the variation in the unit strain on the metal in the flanges will also be proportional to the same areas. Again, the relative importance of the strain at different points in shortening or lengthening the horizontal distance between the two ends of the segment of the arch, according as the curve of equilibrium may be below or above the axis of the rib, will be proportional to the perpendicular distance between the two curves multiplied by the ordinate of the arch at the same point; hence the total lengthening of the horizontal distance between the two ends of the heavily-loaded segment of the arch will be proportional to the contents of the solid *a e f d i* (Fig. 35), of which the sectional area

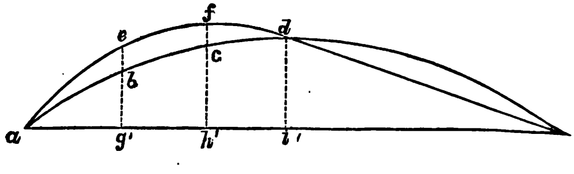


FIG. 35.

at any cross section is equal to the ordinate of the arch at that point, multiplied by the perpendicular distance between the two curves; thus, at the line *g e* it would be equal to $g\,b \times b\,e$, and on *h f* equal to $h\,c \times c\,f$.

It is easy, therefore, to ascertain when the curve of equilibrium is properly placed, since it is only necessary to see that the contents of the solid for the heavily-loaded segment last referred to, is equal to that of the corresponding solid for the lightly-loaded segment. In practice the position of the curve may be very nearly approximated to upon first trial. If it were necessary to attain great exactness there would be some time wasted possibly in shifting the position, but great accuracy is neither essential nor useful.

For, even if the position of the curve were defined to the greatest degree of accuracy attainable by the most elaborate and refined process of mathematical investigation, the final result would after all be based upon the assumption that the arched rib, as a whole, was of uniform elasticity throughout. But every practical man, who has noted the behaviour of iron girders under bending stresses, knows that whilst one girder may deflect a certain amount under the test, another one precisely similar, and placed apparently under precisely the same conditions, may deflect some 30 per cent. more or less. Now we cannot justly assume an arched rib, whether of cast or wrought iron to be more perfectly elastic than an ordinary girder of the same material and construction; consequently, although we may calculate mathematically to any degree of refinement how much one segment of the arch should spread and the other segment close under a certain distribution of the load, it does not necessarily follow that these laboriously-deduced results will after all be within even 30 per cent. of the actual truth, even if the work were put together with more than the average amount of care. But in the simple system of computation advanced in this Section, the spreading and closing of the segment are represented by the contents of certain solids; and it would hardly be possible to make an error amounting even to a tithe of that which might result from imperfect elasticity of the arched rib, for the eye would detect it at once without even scaling. With very little care results may be obtained from the diagram differing less than 5 per cent. from the exact mathematical value of the same; and when the peculiar conditions of the case are remembered, it will be obvious that the former result may be justly considered to be quite as near an approximation to the truth as the latter.

It may be well to illustrate the method of fixing the position of the curve of equilibrium, or the line of equal horizontal thrust, by one other example. The last case taken was that of an arched rib loaded more heavily upon one side than upon the other; the same arched rib will now be taken, with a concentrated load at the centre, and, for the sake of simplicity, the weight of the structure itself will be neglected. Under this load the centre portion of the arched rib will deflect, and become of a flatter radius, whilst the two haunches must of necessity rise, and the radii of the same become sharper. It follows, as explained with reference to the previous example, that the lines of thrust—straight in this instance—pass *above* the axis of the rib at the centre, and *below* it at the haunches, as shown in Fig. 36. In order

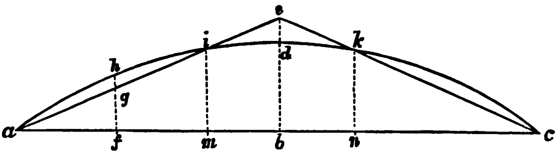


Fig. 36.

to obtain the thrust of the arch, and the values of the bending stresses, it is only necessary to know the height of the summit of the diagram from the base line, *a c*, that is the distance *b e*. Now, as the span, *a c*, is constant, the spreading of the feet of the arched rib due to the excess of compression upon the *top* flange of that member throughout the segment, *i k*, must necessarily be equal to the closing of the same resulting from the excess of compression upon the *bottom* flanges of the segments *a i* and *k c*. The relative values of the spreading and closing elements are represented respectively by the contents of the solid, *m i e k n*, and of the two solids, *a i m* and *c k n*; the cross section of either at any point is equal to the ordinate of the arched rib at the same point multiplied by the perpendicular distance between the line of thrust and the axis of the rib; so that at the line *f h* the section $= f\,h \times g\,h$, and at $b\,e = b\,d \times d\,e$. To fix the position of the line of thrust, then, it is necessary to assume some position for the point *e*, and to test its accuracy by estimating the contents of the solids. Upon trial it will be found that with a parabolic arched rib the distance *b e* is about

1.28 times the rise of the arch. Hence, W being the weight in tons applied at the centre, S the span, and r the rise of the arch in feet, the thrust (T) in tons of an arched rib of uniform cross section, and pivoted at the ends, will be:

$$T = \frac{WS}{1.28 \times 4r}.$$

Again, the distance *d e* being about .28 of the rise, and *g h* about .13 of the same, the maximum bending moment occurring at the centre of the span will be .28 T r, and at the haunch .13 T r. But the value of T varies inversely as the rise of the arch, hence the moments of the bending forces will be dependant upon the intensity of the load and upon the span alone; or, in other words, the strength of the arched rib as a girder must be the same whatever the rise may be. Substituting the value of T in the last expressions, the bending moment at the centre of the span will be $M = .0548\,W\,S$; and at the haunch $M' = .0253\,W\,S$.

The bending stresses may also be obtained from the diagram by the following procedure: Referring to Fig. 36, it will be seen that the line of thrust corresponds with the axis of the rib at two points, *i* and *k*, and it follows that the rib might be severed at those points without affecting the conditions. The rib, therefore, may be considered to be made up of three segments, the centre one, *i k*, and the two side ones, *a i* and *k c*. Now, the form of the arch being parabolic, it follows that an uniformly distributed load would be necessary to ensure equilibrium if no provision were made for bending stresses; and the thrust being given, we know that the intensity of the distributed load in tons per foot run (w) would be:

$$w=\frac{8\mathrm{T}r}{\mathrm{S}^2}.$$

Substituting the ascertained value of T, we obtain:

$$w=\frac{2\mathrm{W}}{1.28\mathrm{S}}.$$

As this intensity of distributed load is necessary to keep the arch in equilibrium, it follows that the absence of it will induce a bending stress upon that member of corresponding intensity. Thus, the bending stress at the haunch will be the same *amount* as that occurring upon a straight girder of the span *a m* loaded uniformly with the weight *w* in tons per foot run, but of course opposite in direction; and the bending stress at the centre of the span will be that due to the weight, W, acting at the centre of the span *i k*, *minus* the amount due to the distributed force, *w*, taking effect in a reverse direction upon the same segment of the arched rib.

To obtain the maximum bending moments, therefore, it is only necessary to ascertain the lengths of the several segments into which the arched rib is effectively divided. Now the distributed force due to the reaction of the unloaded arched rib on the length *f b* will be equal to one half of the weight at the centre; hence we have obviously $f\,b=\frac{\mathrm{W}}{2w}$, and by substitution $f\,b=.32\ \mathrm{S}$. It follows that $a\ m=2(.5-.32)\ \mathrm{S}=.36\ \mathrm{S}$; and $i\ k=2(.5-.36)\ \mathrm{S}=.28\,\mathrm{S}$. The maximum bending moment occurring on the segment *a m*, that is, at the haunch of the arch, will be:

$$M' = \frac{(.36S)^2 w}{8} = .0253WS;$$

and on the segment *i k*, at the centre of the arch,

$$M = \frac{.28SW}{4} - \frac{(.28S)^2 w}{8} = .0548WS.$$

These results are, of course, in both instances, identical with those arrived at by the more simple process of obtaining the bending moments direct from the diagram, by taking the product of the thrust of the arch into the perpendicular distance between its axis and the line of thrust. It must not be forgotten that they are subject to modification on account of change of form, or of any initial strain which may be brought upon the arched rib during erection, both of which elements are reserved for consideration in another Section.

In each of the preceding cases the arched rib has been assumed to be pivoted at the ends. Although this system is sometimes employed on the Continent, the plan usually adopted, and which is the most generally advantageous, is the securing of the rib ends firmly to iron skewbacks built into the abutments. The new conditions thus introduced complicate considerably any rigidly mathematical investigation of the strains occurring upon the arched rib; but, as the uncertain elements taking effect in the cases already considered operate with still greater force in the present instance, great exactness in the results is of no real practical utility; and by the method of deducing the strains from a diagram, sufficient accuracy may be attained with very little more labour than was involved in the example already illustrated. Now, when the ends of the arched rib are fixed to the abutments it is obvious that the planes of the two ends will always include the same angle, however much the arched rib may be distorted at other points; or, as it may be otherwise put, the deflection curve at the springing will always be normal to the skewback. This condition, together with the one that the horizontal distance between the ends of the rib is a constant quantity, enables us to fix the correct position of the curve of equilibrium, or line of thrust, with respect to the axis of the arched rib.

For, in order that the span may remain constant under any distribution of the load, the contents of certain solids already referred to, bounded on one side by the ordinates of the arched rib, and on the other by the distance between the line of thrust and the axis of the rib, must balance; and in order that the planes of the two ends of the rib may include the same angle, it is necessary that the total *extension* of each flange of the rib should be equal to the total *compression* induced by the strain upon the same members. But the extension and compression of the flange will be proportional to the total amounts of the corresponding strains, and those amounts it has already been shown are proportional to the areas of the spaces included between the line of thrust and the axis of the arched rib. It has also been shown that the strain is compressive upon the top flange, and tensile upon the lower flange, when the line of thrust passes *above* the axis of the arched rib; and that when the line falls *below* the axis those conditions are reversed. Hence, in order to fix the position of the line of thrust, or curve of equilibrium, with respect to the axis of an arched rib of uniform cross section, it is necessary in the first place to assume some position for it, and then to test the correctness of that position by scaling and computing the areas of the spaces, and the contents of the solids, already referred to.

Thus, taking the arched rib, loaded at the centre as before, but fixed at the ends instead of being rounded, there will be no difficulty in fixing the approximate position of the line of thrust. For it is at once apparent that the centre segment of the rib will be depressed under the load, and that the adjacent segments must necessarily rise and fall alternately; and it has been shown that sinking of the rib at any segment indicates that the line of thrust passes *above* the axis of the rib throughout that segment, and on the other hand that the rising of a segment shows the line of thrust to be *below* it. In the case of the arched rib loaded at the centre, then, it is known that the line of thrust must pass above the axis of the rib at the centre and at each springing, and below it at each branch, as shown on Fig. 37. The line of thrust, *f e i*, is correctly placed on that diagram; that is to say, the area of the shaded portions *above* the axis of the rib is equal to the area of

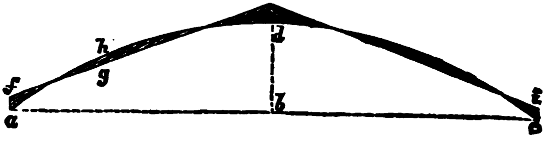


Fig. 37.

the portions *below* the rib, and the product of those areas into the ordinates of the arched rib at the same points also balance.

The rise of the line of thrust is by scale 1.06r, hence the horizontal thrust will be:

$$T = \frac{WS}{4.24r}.$$

Again, the distance $de = .2r$, $hg = .09r$, and $fa = .14r$, and the bending moments will be equal to the products of those amounts into the thrust. Hence the bending moments occurring on a parabolic arched rib fixed at the ends and loaded at the centre, will be .047 WS at the centre; .0215 WS at the haunch; and .033 WS at the springing. The bending moments upon the same rib with rounded ends, it may be remembered, were .0548 WS at the centre, and .0253 WS at the haunch.

The case of the arched rib loaded at the centre has been selected for illustration for the sake of simplicity and brevity; but the same method may be adopted, and in practice little more labour would be involved, in the determination of the strain upon an arched rib of any form and under any distribution of the load. Fig. 38

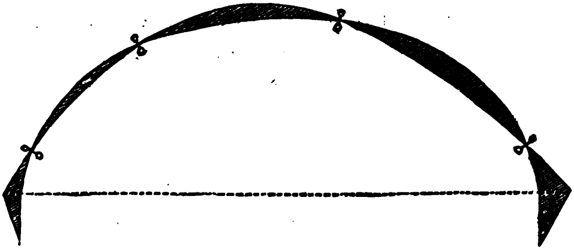


FIG. 38.

shows the application of the curve of equilibrium for an unequally distributed load to the elliptical arch ribs fixed at the ends, adopted for the roofs on the Metropolitan Railway. It is unnecessary, however, to further illustrate this branch of our inquiry.

A general consideration of the results deduced up to the present stage will make it evident that when the form of the arched rib is similar for all spans, and when the distribution of the load is also similar, certain constant relationships will exist between the proportions of the structure and the intensity of the load. Thus, with a parabolic, or what is the same in practice a segmental arched rib, the *direct* compression due to the thrust will vary inversely as the *rise* of the arch, whilst the compression arising from the bending moments will vary inversely as the *depth* of the rib simply, irrespective of the rise. For instance, the direct compression at the crown of the arch will be the product of the distributed load into the span divided by x times the rise of the arch, and the additional compression upon the top and bottom flanges of the rib under an unequal distribution of the rolling load will be the product of the load per foot into the span divided by y times the depth of the rib; and the values of x and y will be constant for all arches of similar form. The necessary diagrams for the determination of the values of these constants have been constructed by the author, and the following rules for the strains upon segmental arched ribs loaded uniformly, or with a rolling of uniform intensity per foot run, have been deduced. It must not be forgotten that in the present Section the strains resulting from change of form have been purposely ignored.

Let W be the total load in tons, dead and rolling, uniformly distributed; r the ratio of the span to the rise (usually$=10$); then the maximum thrust of the arch will be:

$$T=\frac{Wr}{8},$$

and the direct compression upon the arched rib at the springing will be:

$$=T\sqrt{1+\frac{16}{r^2}}.$$

The sectional area of metal in square inches (A) required at the springing when the load is entirely dead, will, consequently, be:

$$A=\frac{T}{t}\sqrt{1+\frac{16}{r^2}};$$

t being the allowed strain in tons per square inch. When a proportion of the load is rolling, the sectional area of the rib will be greater. Let w be the ratio of the total load to the rolling load; and d the ratio of the span to the depth of the rib (usually$=50$); then, for a rib with rounded ends, the area A at the springing will have to be multiplied by

$$\frac{\frac{d}{4r}+\frac{3}{4}+w-1}{w}$$

for that at the haunch; and the sectional area of the rib at the centre may be made the mean between the two areas. For a rib with fixed ends the area A, required at the springing when there is no rolling load, will have to be multiplied by

$$\frac{\frac{d}{5r} + \frac{3}{4} + w - 1}{w}$$

for the area at the same point when a portion of the load is rolling; and the area at the centre of the rib may be made the mean between the latter area and the value of A.

When the sectional area of the web of the arched rib forms a considerable proportion of the whole, as in cast-iron bridges, the gross sectional area should be somewhat greater than the preceding formulæ indicate. The necessary corrections may be effected when designing the cross section of the rib by assuming the *effective* area of the web to be the actual area divided by $1 + \frac{3}{w}$

We may now proceed to the consideration of the modifications required to be effected in the results arrived at in the present Section, on account of the disturbing influences arising from the change of form of the arched rib, and differences of temperature.

Section 26.

ON THE STRAINS DUE TO CHANGE OF FORM.

The change of form resulting from the imposition of a load upon an arched rib of symmetrical cross section, and subjected to no initial stress, is such as to augment, in every instance, the unit strain upon the material, beyond the amount which would obtain if the arch maintained its normal form. Thus, it has been shown that if the line of thrust passes *above* the axis of the arched rib the curvature of that member will be flattened, and consequently the distance between the line of thrust and the axis of the rib will be increased by the amount of this deflection, and the bending moment will necessarily also be increased *pari passu.* Analogous effects result from the cambering of the rib, when the line of thrust passes *below* the axis; but, in practice, both these increments of strain may be safely neglected, since within the elastic limit these stresses will form but an inappreciable fraction of the whole, whilst beyond that limit they will be provided for in the corrections hereafter to be considered, when treating on the condition of an arched rib as a long column. The only practically important increase of strain resulting from change of form is that arising from the alteration in the total length of the arched rib, whether arising from the direct compression of the metal under the thrust, or from the expansion and contraction of the same from changes of temperature.

If the length of an arched rib, abutting against two immovable piers, is shortened by any agency, the arch must come down at the crown. The mathematically exact resulting deflection is not easily ascertained, but the practically exact result is very readily determined. Let e = the increment or decrement in the length of the arched rib, and r, as before, the ratio of the span to the rise of arch; then the corresponding rise or fall at the crown will be given very nearly by the equation:

$$\delta=\frac{11re}{40}.$$

Now, the arched rib is not articulated, and consequently the rise and fall of the crown cannot take place without inducing a transverse strain upon that member. A little consideration will make it evident that the resulting strain upon the metal will be practically the same as that occurring upon a straight girder of the same span and depth as the arched rib, and loaded uniformly till the same degree of deflection is attained. It may readily be shown, also, that the arched rib with rounded ends is placed under the same conditions as an independent girder, whilst the arched rib fixed at the ends is analogous to a continuous girder of the same span. The strain per square inch under a given deflection will obviously then be dependent upon the span and depth of the arched rib alone, irrespective of its sectional area; hence the required modification in the section of that member on account of the change of length resulting from the compression of the metal, or from changes of temperature, will be most conveniently effected by modifying to the necessary extent the value of the working strain per square inch. Thus, if the strain per square inch on account of changes of form were found to be 1 ton, whilst the maximum working stress was fixed at 4 tons per square inch, the required modification would be effected by substituting $4-1=3$ tons per square inch for the working strain.

Knowing the rise and fall of the arch corresponding to any given increment or decrement in the length $=\frac{11re}{40}$, the resulting strain may be readily deduced from the data afforded by the previous Sections on the deflection of beams. Taking first the strain resulting from the compression of the metal in the arched rib, we have $e = \mathrm{E\,S}f$, when E = extension or compression in terms of the length for each ton per square inch strain, S = span of the arched rib, which, in this instance, may be safely substituted for its length, as the difference between the two, when the rise as usual is $\frac{1}{10}$th of the span, is only some $2\frac{1}{2}$ per cent.; and f the *mean* strain in tons per square inch upon the metal. By substitution, therefore, we obtain:

$$\delta=\frac{11\mathrm{ES}f}{40}.$$

But in a previous Section it has been shown that the deflection of a beam of uniform section under a distributed load is:

$$\delta=\frac{5\mathrm{S}^2\mathrm{EF}}{24\,\mathrm{D}};$$

when F = the maximum strain per square inch tension and compression, and D the depth of the beam; hence,

$$\mathrm{F}=\frac{24\mathrm{D}\delta}{5\mathrm{S}^2\mathrm{E}}.$$

Substituting the value of

$$\delta=\frac{11\mathrm{ES}f}{40},$$

and also $d=\frac{\mathrm{S}}{\mathrm{D}}$ for ratio of span to rise as in last Section, we obtain the maximum additional strain in tons per square inch (F), resulting from the compression of an arched rib of uniform section and rounded at the ends, under a mean strain of f tons per square inch, as follows:

$$\mathrm{F}=\frac{4rf}{3d}.$$

The quantity E does not appear in this expression; hence the strain per square inch will be the same for cast iron, wrought iron, and steel arched ribs.

For a square-ended rib, allowing for imperfect fixing, by taking the modulus of elasticity at double its actual value as far as the end portions are concerned, the points of contrary flexure will occur at a distance from the piers equal to $\frac{1}{6}$th of the span. Hence, F being the maximum strain per square inch on the metal, which will, of course, take effect at the springing, we have:

$$\delta=\frac{\mathrm{S}^2\mathrm{EF}}{9\mathrm{D}};$$

and consequently by substitution, as in the instance of the round-ended rib, we have

$$\mathrm{F}=\frac{5rf}{2d}.$$

At the centre of the span the value of F will be $\frac{4}{5}$ths of the amount given by the preceding equation. The strains in tons per square inch, resulting from the compression of the metal in arched ribs which have not been subjected to any initial stress, will, therefore, be given by the following expressions:

Round-ended arched ribs. Strain at centre $F=\frac{4rf}{3d}$

Square " " " " $F=\frac{2rf}{d}$

Ditto " " " springing $F=\frac{5rf}{2d}$.

As before observed, the value of F will be the same for all symmetrical cross sections, irrespective of the actual sectional areas; consequently, t being the allowed working strain in tons per square inch, it will only be necessary to take the amended value of $t=t-F$, in order to provide for the strains incidental to the change of form resulting from compression of the metal in an arched rib of symmetrical cross section. If the rib, however, is to have flanges of unequal area, the gross sectional area of the metal may be the same as if no change of form occurred. Thus, with a round-ended rib, the area of the flange required on the hypothesis of incompressible material, would have to be multiplied by $\frac{t+F}{t}$ for the area of the top flange, and by $\frac{t-F}{t}$ for that of the lower flange. The same corrections would apply to the centre of a square-ended rib, whilst at the springing the former expression would give the correct area for the *bottom* flange of the rib, and the latter that for the *top* flange.

It appears, then, that deflection merely throws the same amount of stress in unequal proportions on the two flanges of the rib. Now, instead of making these of unequal areas, for practical reasons, it will be generally preferable to alter the normal position of the centre of pressure, by putting an initial strain upon the flanges. The most convenient way of effecting this is either to make the rib originally to fit a span a certain amount larger than the actual one, and to spring it into its place, or else to cotter the rib up after it has taken its bearing. The excess of span, or amount of cottering up required, will obviously be f S E; an amount which will be sufficiently nearly, and more conveniently expressed by the equation: Excess of span in inches,

$$=\frac{TS(w-1)}{800wa}$$

for wrought iron; 1.7 times that amount for cast iron; and $\frac{4}{5}$ths of the same for steel arched ribs, either fixed or rounded at the ends. We may now proceed to consider the analogous case of elongation and contraction of an arched rib from change of temperature.

The observed movement of iron bridges from change of temperature is generally considerably less than might be inferred from calculation; partly owing, no doubt, to the very gradual transition from one extreme to the other in most countries, and partly to imperfect freedom of movement. Thus, $\frac{1}{2}''$ in 100 ft. is about the maximum variation in the length of iron structures in this country, and, since the ascertained elongation of a simple bar between the freezing and boiling points is $\frac{1}{800}$th of its length, the preceding amount would appear to indicate an effective range of 60° only in the temperature. Assuming the ironwork to be fixed at a mean temperature, it is only necessary then, to provide for a movement of $\frac{1}{4}''$ in 100 ft. Substituting that value of e in the equation,

$$\delta=\frac{11re}{40},$$

we have the rise and fall in inches (δ) of the arched rib under extremes of temperature:

$$\delta=\frac{11rs}{16000}.$$

If r as usual$=10$, the preceding expression shows the rise and fall of the arch beyond its normal position at the mean temperature, to be $\frac{11}{16}$th of an inch for each 100 ft. span, or a total range, of course, of $1\frac{3}{8}''$. The resulting strain in tons per square inch (F'), determined in the manner already exhibited when treating on the strains resulting from compression, will be:

$$F'=\frac{r}{3630Ed},$$

for a round-ended rib at the centre: $1\frac{1}{2}$ times that amount for a square-ended rib at the same point, and $1\frac{7}{8}$ times at the springing. Substituting $E=.00017$ for cast iron; $.0001$ for wrought iron, and $.00008$ for steel, the expressions for the several strains become as follows:

	Round-ended rib. Centre.	Square-ended rib. Centre.	Ditto springing.
Cast iron	$F'=1.62\frac{r}{d}$	$F'=2.42\frac{r}{d}$	$F'=3.02\frac{r}{d}$
Wrought iron	„ 2.75 „	„ 4.12 „	„ 5.15 „
Steel	„ 3.44 „	„ 5.16 „	„ 6.45 „

The value of F' is the same for all symmetrical cross sections; therefore, in order to include the strains arising from changes of temperature, it is only necessary to substitute for the working strain in tons per square inch t, the value $t-F'$. Should the range of expansion be greater than $\frac{1}{2}''$ in 100 ft., the value of F' would be increased in the same ratio; thus, for American bridges, it would be necessary to take F' at about double the amount given in the preceding Table.

Having now examined the strains on a simple arched rib, we are in a position to consider what modification of form, if any, would enable the arch to do its work in a more efficient manner. In the first place, a distributing girder along the top of the arch, sharing all the disturbing forces with it, would be very inferior in effect to the same amount of additional metal thrown into the rib; indeed, one of the first principles of economic design is concentration, not division of load. The same objection applies, in a less degree, to a distributing girder dying away in the arch at the crown. In neither case would it be advisable to assume any assistance from the girder as a strut to the abutment. Unless the rolling load were proportionally great and the depth of rib unusually small, under certain conditions of temperature, the arch would be doing all the work before the strut could come into play. Under any circumstances, one end only of the girder should be fastened to the abutment, the other being free for movement. If the spandril filling between the arch rib and the horizontal girder were capable of acting as bracing, from the increased depth of girder thereby obtained, the strain at the haunch would be lessened, but, for the same reason, that at the centre would be increased. The balance of advantages would rest with one side or the other, according as the rolling load was heavy or light. The increased strain at the centre, evidently arising from the imperfect freedom for movement at the crown of arch, naturally suggests dividing the rib at that point, and rounding each of the ends. This has been done in several German bridges. With adequate depth of bracing, the horizontal girder and arch rib being, say, one-fifth the rise of arch from centre to centre at the crown, the maximum strain on the rib would be the same as if no disturbing forces existed, and the maximum on the girder would be $T \frac{7}{16w}$. Since this arrangement meets all the requirements of the problem as previously stated, namely, adequate depth of girder and the elimination of all strains from deflection or change of temperature, it is unnecessary here to consider other modifications of the single-arch bridge. Though other solutions of the problem are possible, they could not present any greater theoretic advantage; the only improvement, therefore, would be one of detail, affecting chiefly special cases.

It is necessary now to consider the condition of an arch when the abutments are more or less movable, or, practically, when it is one of a series in a viaduct; for although the end abutments may be safely assumed to be immovable, as was done in the case of the single-arch bridge, the intermediate piers are rarely, if ever, equally stable. Very little consideration will make it apparent that the great difference between the two cases is in the amount of strain caused by the rolling load. As far as the equally distributed load is concerned, the *natural* thrust of all the arches being equal, the stability of the intermediate piers is a matter of indifference. The same thing may be said with regard to the strains from deflection and change of temperature. It is otherwise, however, when the rolling load is concerned; in that case, as the thrusts of adjacent arches do not always balance, the difference must be taken up either by the piers or by a modified artificial thrust on the whole series of arches. The most disadvantageous conditions will be obtained when the intermediate piers have *no* stability, the neighbouring arches being the only abutment. Under these circumstances, the amount yielded by the abutments, and therefore the strain, will be governed, to a great extent, by the number of spans in the viaduct. The desirable amount of adjustment for deflection will also be affected by the same circumstances; though smaller in amount than before, it will, owing to the greater depth of arched rib usually obtained in this case, be of at least equal importance. The amount of cottering-up in inches suitable for average examples, in order, as before, to admit economically of an equal-flanged cross section of rib, will be about:

$$\frac{\mathrm{TS}(1+wn-n)}{800awn}.$$

The factors already given for the strains from deflection and change of temperature may be used in this instance also without material error, the same reasoning being applicable to both cases.

In estimating the amount of strain from the rolling load, it is unnecessary to consider the effect of any particular distribution of load on an individual span, since it will be insignificant, compared with that resulting from the position of the load on the viaduct as a whole. The maximum stress on the upper flange of any rib takes place when the arch of which it forms a part is loaded, and all the remaining ones are light; the maximum on the lower flange is obtained under the reverse conditions, that is, when the arch under consideration is the only unloaded one. And, since the amount of adjustment given for deflection will approximately bring the metal into a state of equal compression, as far as the thrust is concerned, when the greatest bending moment occurs, the maximum strain may be taken at the same amount on both flanges. If the ends of the ribs be rounded, the area at the springing will be the same as that previously given for the single-arch bridge, under similar conditions, which amount should be gradually increased to

$$\frac{T}{t}\left\{\frac{(2d-r)\ (n-1)}{wnr}+1\right\}$$

for the area at the centre of the arch, at which point the maximum stress takes place. In the case of a viaduct in which the ends of the arched ribs are square, and the abutting ones riveted together, the determination of the exact strain involves so many assumptions, and problems of such extreme perplexity, that it would be a mere coincidence if two separate attempts at solution gave the same results. For practical purposes, sufficient margin will be allowed for all *probable* strains, if the area of the rib at the springing be made the mean between that previously given for the springing of the square-ended single-arch rib and the centre of the viaduct rib. The area at the centre may be the mean between the area thus obtained and the area at the same point of the round-ended viaduct arch before alluded to.

Comparing these areas with those required in a single-arch bridge doing the same work, it is at once apparent that in the case of a viaduct with unstable intermediate piers the simple arched rib is not well adapted to the requirements of the rolling load; and that, for economic reasons, some considerable modification of form is desirable. The only advantage in placing a horizontal girder along the top of the arch is, that a greater depth may be given to it, without spoiling the appearance of the bridge, than could be done to the arched rib itself; but it is open to all the objections previously urged against it. And in addition, if the depth of the two members be different, another element of uncertainty, the degree to which they may be relied upon as working together, is introduced. Again, strains from change of temperature, due to greater depth, would be imposed upon the rib, without the corresponding advantage of greater ability to carry the rolling load being obtained.

Continuity of girder is objectionable, since to prevent undue strain on it, freedom for expansion must be allowed at the ends; and if the ends be free, a great amount of strain will be concentrated on the end arched ribs. However, this objection would not be of much weight if the girders were continuous over two spans only; but, at the best, the advantage of continuity in this instance is very questionable, for the reduction in the maximum strain at the centre, under certain conditions of load, would not be equivalent to the increase at the pier under other conditions. The effect of braced-spandril fillings is the same as in the previous case, that is, the rib and girder would require greater section at the centre, and less elsewhere.

The direction in which capability of improvement really lies is evident, since two necessary conditions for economical construction are wanting in the simple arched rib; namely, adequate depth of girder for the rolling load, and freedom of movement for expansion and contraction. Now, with the proportion of rolling load usually obtained in practice, adequate depth for rolling load means the same thing as adequate depth for total load; or, in other words, the considerations fixing the rise of an arch at a certain amount would also fix the depth of the "girder" part at a similar amount. But, since the arched rib itself should be one member of the girder, and since the depth of the girder should be greatest in the centre, it follows that the most effective way of attaining the required stiffness would be by fixing a horizontal girder between the feet of the rib with bracing, in the manner of a bowstring bridge.

The problem would now be solved, were it not for the disturbing influences of deflection and change of temperature, which enforce another modification. Let the arched ribs of *one* end span have rounded ends, and instead of the ends next abutment resting immediately in their shoes, let round-ended struts, in the direction of a tangent to the arch at springing, intervene between them. If this be done, deflection or change of temperature will only cause a slight rise or fall of the end span, not detracting in any way from the efficiency of the arch and bracing. On the contrary, if the rake of the struts be properly adjusted to produce a certain initial compression on the horizontal girders between the feet of the ribs, the maximum strain on those members will be considerably reduced. If these precautions be taken, the greatest area necessary for the arched rib would be the *least* maximum for any arch:

$$\left(\frac{T^2}{t}\sqrt{1+\frac{16}{r^2}}\right),$$

and that for the horizontal girder, with moderate length of strut, should not exceed the fraction,

$$\left(\frac{3}{4w}+\frac{n}{100}\right)$$

of that amount. This last system is, theoretically, the most economical attainable for a viaduct with unstable intermediate piers; it also has the practical advantage of a level soffit. In principle it much resembles the high-level bridge, designed by Mr. Fowler to carry his proposed "outer circle" railway over the Thames.

In the majority of cases, the intermediate piers of a viaduct have a certain amount of stability; the extent to which this may be relied upon, as influencing the strain on the arched ribs, must be, to a great extent, decided by sound judgment in the individual instance. Width of pier is valuable, not only on account of the direct increase of stability thereby obtained, but also of the opportunity it affords of increasing the effect of that stability, and at the same time the advantage of continuity, by fixing the arched ribs and girders firmly to the pier with long holding-down bolts. In all cases, the necessary section of arched rib will be included between those required in the two extreme cases we have considered —the single-arch bridge and the viaduct with unstable intermediate piers; the side to which it inclines may be roughly determined by considering the actual stability of the pier with reference to the required stability.

The result of the preceding investigation affords another illustration of the fact that mechanical and æsthetical perfection are often incompatible. Following the track pointed out by theory, we have arrived at forms of the arch decidedly ugly, compared with the original model. The simple arched rib, with horizontal girder, is an admirable framework for artistic treatment; if this be taken advantage of, it forms a bridge at the same time elegant and expressive of the work it has to do. But, unfortunately, in cases when economy is the first consideration, it is seldom admissible. One thing is clear: if this system be adopted, it should be treated in a manner expressive of the fact that the alternative of mechanical superiority has been duly considered and overruled. Thus the spandrils of the arch, so often crammed in with a lot of zigzags expressing nothing but feebleness of conception, should form one of the most artistic features of the design; indeed, there are so many constructions to which the skull and crossbones pattern of bracing is essential, that it may well be dispensed with in spaces so utterly unfitted for it as the spandrils of ordinary arch bridges.

Nothing now remains but to consider briefly the conditions of the arched rib as a long column. It is commonly assumed in investigations relating to the strength of arched ribs that, if the mathematical position of the centre of pressure corresponds with the centre of gravity of each cross section, the unit strain will be of uniform intensity over each cross section, no matter what proportion the length of the rib may hold to its depth. Now, although no direct experiments have been instituted to determine the resistance of arched bars to compression induced by a uniformly distributed load, the results of experiments on the resistance of straight bars to compression afford conclusive evidence of the fallacy of such an assumption. For, if we may assume the stress to be uniformly distributed over the cross section of an *arched* bar when the line of thrust is coincident with its axis, we are equally justified in assuming the same conclusion to hold good with a *straight* bar; and on the same hypothesis it might appear that a round-ended column would be stronger than a square-ended one, by reason of the pressure being more certainly applied in the direction of its axis. But, in a previous Section, it has been shown that both theory and experiment indicate the resistance of a long column to be governed, not by the resistance of the material to direct compression, but by its degree of elastic reaction; that is to say, the resistance of a long column is represented by the end pressure required to maintain it in a bent form, and it has been shown that this pressure will be the same for all degrees of flexure within the elastic limit. In the same way the resistance of a shallow unbraced arched rib will not be dependant solely upon the crushing resistance of the material, but upon the amount of end pressure necessary to keep a segment of the rib *of a certain length* in a bent form. Now, with a round-ended straight column the length of the segment to be maintained in a bent position is equal to the entire length of the column; whilst in a square-ended column the length of the equivalent segment has been shown to be equal to the length of the column divided by $\sqrt{3}$, which corresponds with the distance between the two points of contrary flexure in such a column. But an arched bar with rounded ends may be distorted with but one resulting point of contrary flexure—at the centre of the span; hence, it may be considered to be made up of two columns rounded at the ends, equal in length to one half of the span, and the resistance of the arched bar cannot justly be estimated at a higher amount than the end pressure necessary to maintain such columns in a bent form. The Tables of the resistances of columns, given in Section 20, are for square-ended columns; consequently it will be convenient to reduce the preceding round-ended segments to the equivalent ratios if square-ended. This ratio (r') will evidently be given by the equation:

$$r' = \frac{d}{2}\sqrt{3} = .866d;$$

d, as before, being the ratio of span to depth of rib.

Proceeding in the same way with square-ended arched bars, since the bar cannot be distorted without three points of contrary flexure being formed, it may be conceived to be made up of two columns similar to the last, but fixed at *one* end, hence the equivalent ratio,

$$r' = \frac{d}{2}\sqrt{2} = .707d.$$

Having thus deduced the equivalent ratios, the resistance of an arched bar, as far as it can be determined without direct experiment, may be obtained by inspection from the Tables previously referred to. Since arched *ribs* are universally made of a girder section, the equivalent ratio will be smaller. The transverse strength per square inch of an average arched rib section may be taken at three times that of a solid round bar of the same depth; hence, the value of r' for ordinary arched ribs will be as follows:

Round-ended rib $r'=.866d\sqrt{3}=.5d.$

Square " $r'=.707d\sqrt{3}=.4d.$

The practical effect of the preceding deduction will be best illustrated by a specific example, thus: Given a parabolic wrought-iron square-ended arched rib uniformly loaded, and adjusted to neutralise the compression of the material, what would be the resistance per square inch, assuming the rib to be unassisted by any external bracing, and the ratio of span to depth $(d)=50$, as usual? Here the equivalent ratio $r'=50\times.4=20$, and by the Table for round wrought-iron columns the corresponding resistance is 284 cwt. per square inch; or the loss of strength in the arched rib, under the given conditions, will be about $\frac{1}{5}$th of the stress it would sustain if efficiently braced. Where the arched rib is shallow and unassisted by external bracing, as in the instance of the proposed St. Louis 515 ft. span steel arch bridge, this element of diminished resistance is too considerable to be justifiably neglected.

The conclusions thus arrived at are perfectly consistent with Mr. Fairbairn's deductions from his valuable series of experiments on the resistance of tubes to collapse. The lengths of the tubes subjected to test were, unfortunately for our purpose, too short to allow of the assistance afforded in the diaphragms at the ends being eliminated, and the resistance of the tube *per se* ascertained. If due allowance be made for this disturbing element, the results of the experiments corroborate our conclusions. Referring, for example, to the largest scale experiment cited by Mr. Fairbairn, that of the boiler for the London and North-Western Railway Company, 7 ft. diameter, by 35 ft. long, with internal tubes 3′ 6″ diameter, and $\frac{3}{8}''$ thick; let it be required to ascertain the collapsing pressure per square inch on the hypothesis advanced in this Section, assuming the tube to be unassisted by the end plates of the boiler. Now, in distorting a circular tube there must be four points of contrary flexure, and the equivalent length of column rounded at the end, therefore, will be $\frac{D\pi}{4}$. The strain per square inch the metal of the tube may be expected to sustain will, consequently, be equal to the strength (F) of such a column of square construction. Now, P being the required collapsing pressure in lbs. per square inch, k the thickness of the tube, and D its diameter in inches, we have

$$\frac{PD}{2} = Fk;$$

hence, the formula for the resistance in *cwt.* per square inch, of *round* columns fixed at the ends being,

$$B.W. = \frac{340}{1 + \frac{r^2}{2000}},$$

we have, by substitution,

$$P = \frac{76000k}{D + \frac{D^3}{2000k^2}}.$$

Substituting $k = \frac{3}{8}''$ and $D = 42''$, we have the calculated collapsing pressure of the given tube, P = 93 lbs. per square inch. By actual test the pressure was 97 lbs. per square inch. This inappreciable difference between the calculated and experimental pressures clearly proves the analogy existing between the two extremes of the series of columns—the straight and the annular. We may, therefore, confidently assume that the same laws will govern the resistance of columns included in the intermediate links of that series, that is to say, of segmental columns or arched ribs.

We have now considered the conditions of an arched rib under the several heads of the stresses due to the distribution and intensity of the load; of change of form resulting from the compression of the material, to be neutralised by a certain preliminary adjustment; of the effects of temperature; and finally, of the action of the arched rib as a long column.

Before entering upon the consideration of the problems relating to the resistance of arched ribs, it was maintained that extreme accuracy in the deduced results must necessarily be apparent only; since the hypotheses upon which those results were based are, after all, in no instance *certain* to be attained. The exercise of sound judgment when framing an hypothesis is always of great moment, whilst minute accuracy in the final deductions from the same is very rarely of any practical use to the engineer; hence, the most generally advantageous method of investigation is that affording the greatest degree of simplicity. This consideration has been kept well in view in the preceding investigations, and it will have been observed that useless refinements have, in all instances, been sacrificed to attain simplicity. That such a procedure does not necessarily involve any important practical difference in the final results, as compared with those arrived at by the most laborious and elaborate methods of investigation possible, will, we think, be satisfactorily proved by the following comparison of the results given by the formulæ advanced in this Section, with those advanced by Captain Eads, in his able Report on the proposed St. Louis steel arched bridge. The final results were arrived at only after "several months of patient labour," by Messrs. Flad and Pffeifer; and certainly, the eighty closely-printed pages, filled with formulæ and condensed calculations, forming a moiety of Captain Eads' Report, afford satisfactory evidence that no time or trouble was spared to ensure mathematical exactness in the results. The proportions of one of the arched ribs of the St. Louis bridge, as finally settled, are as follows: Span, 515 ft.; rise, 51′ 6″, or $\frac{1}{10}$th of the span; depth of rib, 8 ft., or $\frac{1}{64}$th of the span, fixed at the ends; sectional area for a distance of $\frac{1}{12}$th the span from each springing, 216 square inches; sectional area at centre and throughout remainder of arched rib, 144 square inches, no extraneous bracing. The load to be carried by each of these ribs is made up of 515 American tons dead, and 412 tons live load, and the allowed working strain is 12½ American tons per square inch. We will now calculate the sectional area required by the formulæ derived from the preceding investigation. Referring to the formulæ for arched ribs in Part 7, we have the sectional area at springing in square inches,

$$A = \frac{ac}{t - \frac{15rx}{8d}};$$

and at centre,

$$A'=\frac{a(c+1)}{2t-\frac{3rx}{d}};$$

when

$$a=\frac{Wr}{8}\sqrt{1+\frac{16}{r2}};$$

$$c=\frac{\frac{d}{5r}+\frac{3}{4}+w-1}{w};$$

$x=9$ for steel bridges in America; $r=10$; $d=64$; $W=927$ American tons; $t=12.5$ American tons; and $w=2.25$; hence, by substitution, we have

$$a=\frac{927\times 10}{8}\sqrt{1+\frac{16}{100}}=1250;$$

$$c=\frac{\frac{64}{5\times 10}+\frac{3}{4}+2.25-1}{2.25}=1.46;$$

hence, the area at springing,

$$A=\frac{1250\times 1.46}{2.5-\frac{15\times 10\times 9}{8\times 64}}=186 \text{ square inches};$$

and the area at centre,

$$A'=\frac{1250\times 2.46}{25-\frac{3\times 10\times 9}{64}}=148 \text{ square inches.}$$

Comparing these areas with those adopted by Captain Eads, it will be seen that the total quantity of steel is practically identical in both instances, but it is slightly differently distributed. Thus, the toe of the rib in Captain Eads' bridge is, for a length of some 10 or 15 ft., of greater sectional area than is necessary according to our formulæ; whilst, on the other hand, the remainder of the 515 ft. span is some 3 per cent. smaller. The reason why we prefer the distribution of metal indicated by the formulæ are the following:—In the St. Louis bridge no allowance at all was made for imperfect fixing of the ribs, whilst it may be remembered, to include that contingency, we effected the necessary correction roughly, by taking the modulus of elasticity of the material at the toe of the rib at one half the usual amount. However carefully the skewbacks might be bedded when an excess of pressure occurred at one of the edges, it is pretty certain that the masonry would to a slight extent yield under the more intense unit pressure, and thus tend to equalise the same. Even if no actual compression occurred at the joints the natural elasticity of the stonework would tend to effect the same modification in the strains on the arched rib, and for that reason we should prefer giving the 4 square inches additional sectional area throughout the main portion of the rib, and the smaller amount immediately at each toe, as indicated by our formulæ.

The maximum effective value of the unit stress upon the steel, it is maintained by us, will be greater than the assumed working strain of $12\frac{1}{2}$ tons per square inch, on account of the uneven distribution of the stress arising from unequal deflection, and the action of the arched rib as a long column. The probable *ultimate* resistance of the metal in an arched rib has been shown to be identical with that of a round column of the ratio $.4r$, which in the instance of the St. Louis bridge has the value of 25; hence the ultimate strength of the steel in the arched ribs should be derived from the compressive resistance of a bar, say 1″ diam. and 25″ long.

Referring to the Table of steel columns, it will be seen that the ultimate strength of such a bar of strong steel is 602 cwt. per square inch; hence, reducing the 12½ American tons the cwt., it would appear that a load somewhat less than three times the actual load would suffice to bring down the bridge. But, on the other hand, the unit strain brought upon the steel by the actual load would not exceed ⅕th of the stress induced by three times that load; hence the factor of safety obtained in the St. Louis bridge might be given at 3 or 5, according as the ultimate breaking load of the arch ribs, or the ultimate unit resistance of the material of which they are composed, were taken as the standard of measurement. We have here an illustration of our proposition that "working strain = ⅕th ultimate unit resistance, and working load = ⅕th breaking load are anything but synonymous terms." In a large bridge a sufficient margin of strentgh will no doubt be provided if the working load do not exceed ⅓rd of the breaking load, and at the same time the working unit strain do not exceed ⅕th of the ultimate resistance of the material. In a small bridge it would generally be more satisfactory to keep the working load within ¼th of the breaking load, although the working unit strain might remain ⅕th of the ultimate, as before.

It will readily be granted that formulæ which will include such an exceptional case as the St. Louis bridge are sufficiently comprehensive, and reliable for all practical purposes. It is unnecessary, therefore, to test further the accuracy of the results given by the formulæ, but we will illustrate the application of the same in one other instance—that of Mr. Fowler's Victoria Bridge over the Thames at Pimlico—the mathematical value of the probable strains upon which were calculated to a considerable degree of accuracy. The dimensions of one of the arched ribs is as follows:—Span 175 ft.; rise 17′ 6″ or $\frac{1}{10}$th of the span; depth of rib 3′ 6″ or $\frac{1}{50}$th of the span; sectional area of the rib at centre $=82$ square inches; dead load 78 tons, and live load 110 tons. The strength of the arched rib at points beyond the centre is supplemented by that of a strong horizontal girder; so let it be required to know the maximum probable strain in tons per square inch (t) occurring at the centre of the span. By the formulæ we have:

$$t=\frac{a(c+1)}{2\mathrm{A}'}+\frac{3rx}{2d}=\frac{250\times 2.43}{2\times 82''}+\frac{3\times 10\times 2.75}{2\times 50}=4.52 \text{ tons}$$

per square inch; an amount differing less than $\frac{1}{2}$ per cent. from the strain of 4.5 tons per square assumed in the carefully elaborated calculations instituted to determine the requisite strength of this wrought-iron arched bridge.

PART VI.

Section 27.

ON THE PROPER DEPTH FOR GIRDERS.

In designing an iron girder the first step necessary to be taken is to decide upon the advisable depth, because this, in its relation to the span, is the ruling dimension which settles the proper proportions of all parts of the girder. In many cases, limited headway, artistic requirements, or some other extraneous circumstance, necessitates the adoption of some given depth; but in the majority of instances it is entirely optional. In the latter cases, however, as in the former, the depth is—if we view the matter in the proper light—determined for us in an equally definite manner, and can never be justly viewed, as it frequently is, as a comparatively unimportant detail in which we are to be guided chiefly by the prevalent usage. Indeed, the fallacy of accepting the average practice as a correct guide to be followed blindly is remarkably apparent in the present instance: Thus, in 1849, at the Institution of Civil Engineers, Mr. Fairbairn advocated strongly the proportion of $\frac{1}{15}$th the span for the depth of girders up to 150 ft. span; and with the view of keeping the centre of gravity low, and so preventing oscillation, in large girders, he considered the depth should not exceed $\frac{1}{15}$th the span. At the present time the depths of such girders would probably be made one-half greater, and the increased proportion would be given to the longer instead of to the shorter spans. Now, no one will dispute that the best proportioned girder is the one giving the maximum amount of strength with the minimum amount of material; and that being so, a careful consideration of the subject will make it evident that the suitable depth will always lie between certain variable limits, the position and extent of which will be governed partly by the absolute span, but chiefly by the amount of load per foot run to be carried by the girder. In the present article, as space precludes the exhaustive treatment which the subject deserves, it is merely proposed to illustrate the truth of the preceding theorem; and in order to do this, it will be necessary to consider briefly the nature and comparative amounts of the strains on different portions of a girder.

The laws governing the strains on the flanges of a girder are so well defined and easy of application that but little variety of opinion has ever existed as to the required strength of those portions; but the same cannot be said with reference to the web, the whole theory of which remained in a very unsatisfactory state until it was taken up and thoroughly investigated in 1862 by Professor Airy, one of the few men competent to solve the abstruse problems involved. It is, indeed, true that long previous to that date a correct general notion of the nature of the strains was common enough, but there existed no theory by which the actual direction and amount of the strains in the interior of a beam could be expressed and computed numerically. The experiments on the model tube for the Britannia Bridge indicated clearly that diagonal strains, both compressive and tensile, occurred in the webs of the tube; the direction was evidenced by the wave of buckling following an angle of about 45°, and even crossing the vertical stiffeners to assume that angle; and since the portion of web firmly riveted between the top and bottom flanges and the verticals remained always in effect a parallelogram, and as it was impossible for one diagonal of such a parallelogram to be lengthened without shortening the other, it followed necessarily that no tensile strain could have taken place without involving a corresponding compressive strain acting at right angles to it. A due consideration of these facts induced Mr. Stephenson to advocate strenuously the adoption of plate webs in preference to lattice. He considered the theoretical advantage of the former system to be self-evident, since it is only necessary to conceive the bars of a lattice web increased in number till the edges touch, and we get in appearance a plate web divided in its functions, one half of the thickness being devoted to the tensile and the other half to the compressive strains; and, as there is no reason to conclude that tension on a plate in one direction impairs its resistance to a compressional strain operating at right angles, it follows that only one half the section of web would be required if the metal were thrown into the form of a solid plate instead of contiguous bars.

As the same view of the question was taken by Professor Airy, he considered the first step of his investigation should be to prove the theorem, that "whatever be the number and directions of the forces of compression and tension, their combinations may in all cases be represented by the combinations of two forces at right angles, these forces being sometimes both of compression and sometimes both of tension, and generally unequal in magnitude." This being demonstrated, the succeeding investigation is confined to the consideration of two such forces acting at each point of the web from which the strain on any particle in any direction is directly obtainable. Particular attention is called to the special conditions under which the end portions of the web resting on the piers are placed. In all other vertical sections a compressive force taking place in a certain direction is opposed by an equal force of the same nature acting in the reverse direction, and, as the same holds good with reference to the tensile strains, it follows that although any particle of metal may be crushed or torn, there is no great liability to horizontal or vertical displacement. At the end portions, however, these opposing forces do not exist, and consequently there may possibly be a great tendency to movement, both horizontal and vertical. It is, however, proved by Professor Airy that there is no liability to horizontal displacement, whilst the vertical pressure which a horizontal section of the web over the pier has to sustain is equal at the base to one-half the uniform load, and that this force is reduced uniformly from the bottom to the top, so that at half the depth of the girder it amounts to one-fourth only of the distributed load. As our present object is merely to ascertain the most economical depth for a given girder, it is unnecessary to extend our investigations into the nature and amount of the strains on the different members further than is absolutely required to enable us to determine the weights of the two essential portions of a girder—the flanges and the web. For the sake of simplicity, we shall deal with a distributed dead and rolling load only, although of course there is no difficulty in applying the same reasoning to other arrangements of the load.

Now, the maximum strain on each flange of a girder of uniform depth is, by the simple principles of leverage, equal to the distributed load multiplied by the span and divided by eight times the depth, the strain being greatest at the centre and less elsewhere, in proportion to the ordinates of a parabola. As this strain on the flange must necessarily be transmitted through the web, a little consideration will make it obvious that for a uniform load, whatever the resultant strain may be, the total amount on the half web resolved horizontally must be equal to the maximum horizontal strain on the flange. Or, in other words, confining our attention at present to wrought iron, and assuming the resistance per unit of area to be the same for all descriptions of strains, the horizontal sectional area of a plate web should be equal to double the sectional area of the flange in the centre; and we may simplify the calculation without materially affecting the result by assuming the proper form of the vertical cross section to be rectangular. Again, by the parallelogram of forces, we know that the gross section of all the bars in a lattice web should be greater in the same ratio as the length of a bar exceeds the horizontal distance included between the two ends.

With a rolling load, although the sum of all the horizontal forces on the web is a maximum when the entire load is on the girder, the maximum strain at a given point of the web takes place when the longer of the two segments, into which that point divides the girder, is fully loaded, and the other one unloaded. Should the entire load be rolling, this would necessitate a strength of web at the centre equal to ¼th of that required at the ends; and to provide for the greatest stress at each point, the total horizontal section of the half web must be equal to 1⅙ times the sectional area of the flange at the centre; consequently, the weight of a plate or lattice web is theoretically ⅙th greater for a rolling than for a uniform load.

Now, the weight of a girder is made up, of course, of the joint weights of the two flanges and the web. That of the first portion will be proportional to the mass, obtained by multiplying the sectional area of the flanges at the centre by the span, and by a coefficient representing the relative cubic contents of the flange to the solid so obtained. Theoretically, the value of this coefficient is ⅔, since that is the relative area of a parabola—the form of flange required—to the enclosing parallelogram; the practical value, however, will always be higher than that fraction. Again, the weight of the web will be proportional to the mass obtained by multiplying the same sectional area by a coefficient of a similar kind to the former one—theoretically equal to one—and by the depth of the girder if a plate web, or, if a lattice web, by the square of the length of a bar divided by the horizontal distance, included between its ends. As we have already remarked, in the latter case the force will be greater in the proportion of the length of the bar to the horizontal, whilst the distance over which it acts will also be the length of the bar, and not the depth simply, as in a plate web.

Expressing the preceding results algebraically, d being the depth of the girder, S the span, l the length of a lattice bar, h the horizontal distance from end to end, and x and y the coefficients for the flanges and web respectively, it appears that the weight of a lattice web girder is to that of a plate girder of corresponding depth and strength as $\mathrm{S}x + y\frac{l^2}{h} : \mathrm{S}x + yd$. Should the struts and ties be placed at different angles, it would, of course, be necessary to take each system separately; and should vertical struts be employed, as too frequently is the case, it is quite clear, without any calculation, that the metal so used would theoretically, be merely so much additional load to be carried by the girder. For, as the total horizontal strain on the web is a constant quantity, whatever the description of bracing applied, and as no portion of that strain could be possibly transmitted to the flange by a vertical member, it follows that the work done by the diagonals must necessarily be the same as it would, were no verticals introduced into the construction. The proportional amount of metal so thrown away is easily ascertained, for we know by the principle of the parallelogram of forces that the strains on the vertical and diagonal members will be as their lengths, and the quantity of metal required will, therefore, be as the square of their lengths; consequently the total weight of the web, in this instance, would be equal to:

$$y\frac{l^2+d^2}{h}.$$

It follows, therefore, that if the lattice bars were placed alternately vertical, and at an angle of 45°, the theoretical weight of the web would be to that of a plate web of equal depth and strength as the sum of the squares of the lengths of the two bars is to the square of the depth; that is, taking the depth as the unit of measurement, as $1^2+2(1^2):1^2$, or as $3:1$. It follows also, since the absolute value of $\frac{l^2}{h}$ is proportional to the sectional area of the flange which varies inversely as the depth, that the weight of a lattice web will be the minimum when $\frac{l^2}{hd}$ is the minimum. But $l^2=d^2+h^2$, hence taking $d=1$ as before, we have $\frac{l^2}{hd}=\frac{1}{h}+h$, the value of which will be a minimum when $h=1$, and be then $=2$; or, in other words, the most economical angle at which the bars can be placed is 45°, and the theoretical weight of a lattice web would then be equal to double the weight of a plate web of corresponding strength, which is the result previously arrived at by another method of reasoning. Now, since the depth, d, appears on both sides of the equation, for the web, it is obvious that such portion of a given girder will, within the limits of this investigation, have a constant theoretical weight, whatever the depth may be. And as the weight of the flanges is inversely as the depth, it follows that some theoretical advantage must accrue from every additional increment to the depth. In practice a limit is placed to the influence of these conditions by the variable values of the coefficients x and y. Before we can advance further in our investigations, it will, therefore, be necessary to deduce the values of these coefficients from actual examples of girders under the conditions usually obtained in practice.

The degree to which the actual weight of a girder will approximate to that indicated by theory is affected by a variety of conditions, the most important being the amount of load per foot run; because the greater the weight of the girder per foot run, the smaller will be the percentage of loss caused by such portions of the metal as would be a constant quantity whatever the amount of the load to be carried. It will be convenient to consider first the disturbing influences tending to augment the weight of the flanges, because to that extent the results will be common to both plate and lattice girders.

We have already observed that the proper form of flange for a girder of uniform depth carrying a distributed load is a parabola, and that, consequently, the mass of the flange absolutely required is the sectional area at the centre multiplied by two-thirds the length. But in practice, as we cannot economically employ plates tapering in width or in thickness, we are compelled to form the flanges of a number of parallel plates; the gross thickness of the flange, therefore, will be reduced in successive steps, and the portions of metal outlying the required parabolic outline will be one of the sources of loss. Now, the more numerous the plates, or, in other words, the greater the proportion of load to depth, the greater may be the number of steps, and, consequently, the nearer the approximation to theory. Another source of loss is that arising from the necessary covers and lap of the plates at joints; the amount of this item will be greater in a certain degree as the span, for with the present facilities for rolling long plates it is unnecessary to introduce joints, either in the plates or angle irons of girders not exceeding the length of 30 ft. or 35 ft. over all; but with longer girders, joints must necessarily occur, and, according to the usual practice, the plates must be shorter, and the joints more numerous, in proportion to the greater width of flange required to ensure lateral stability in the higher spans. We think, however, that the system of building up wide flanges with long narrow plates side by side could be more generally adopted with advantage, since it would neutralise the loss arising from short plates. The curved tensional member of the Mayence bridge was constructed in this manner, (under similar circumstances in the Saltash bridge, it will be remembered, Mr. Brunel employed suspension links); and we know of another instance where a 4 ft. flange was formed of two plates 2 ft. wide, alternating with three plates 16 in. wide. Apart from economy, it is obvious that this arrangement is well calculated to ensure uniformity of strength in the tensional member of a girder. Another unavoidable requirement tending to increase the weight of the flanges is the joining of the several portions together by riveting, which involves the employment of a large quantity of otherwise useless metal in the tensional member, besides the weight of the rivets themselves in both flanges. The loss on this score is tolerably constant for all spans and loads, amounting to from 20 to 30 per cent. of the gross weight of the parts in tension; but as the resistance to compression is also some 20 to 30 per cent. less than the resistance to tension, we can eliminate this disturbing element by taking the sectional area of the top flange as the standard, or by assuming the bottom flange to be all effective and the resistance to compression and tension equal per unit of area.

A very slight consideration of the preceding remarks will satisfy us that the most favourable conditions under which a girder can be placed, taking as the measure of economy the percentage of metal required to carry a given load, are when the span is small and the load great; and that the most unfavourable conditions are obtained when the span is great and the load is light. We find, consequently, on deducing the average value of the coefficient x from a number of existing girders of each class, that whilst in the former case it is equal to $\frac{4}{5}$, in the latter it amounts to $1\frac{1}{6}$; we also find the value for a short girder lightly loaded to be $1\frac{1}{20}$, and for a long girder heavily loaded to be $\frac{13}{14}$. It will be remembered that in all these instances the theoretical value of the constant x, assuming the resistance to tension to be equal to that for compression, is $\frac{2}{3}$. Thus far the investigation applies equally to plate and lattice girders; but, in considering the extent to which our theoretical conclusions with regard to the web will be modified by the requirements of practice, it will be necessary to consider each form of construction separately, commencing with the plate web.

The influence of the weight of the web is the most important element in determining the proper depth for a girder, because whilst all the disturbing influences affecting the flanges also affect the web, there is, in addition, another element introduced, namely, the limiting thickness below which the plates may not be reduced. This is never taken at less than $\frac{1}{4}$ in., and in situations not easily accessible for the purpose of painting the thickness should be increased to $\frac{3}{8}$ in.; again, when the load per foot is large, a thicker plate is usually employed. Now the effective horizontal section of a web of uniform thickness, taking, as in the case of the flanges, a reduced strain to compensate for the loss of section through the rivet-holes, will, since the strain increases uniformly from the centre to the ends, and the span, S, is in feet, be equal to $12\ S \times \frac{1}{2} \times \frac{1}{2}$ thickness $= 3\ S \times$ thickness. Hence, as the least thickness of plate is $\frac{1}{4}$ in., the least effective horizontal section of any web will be $\frac{3}{4}$ S; and there can be but a small reduction in the weight of a web of uniform thickness, whatever may be the lightness of the load. Appealing again to actual examples to decide the amount of the variation, we find the gross average section, including stiffeners, covers, and rivets, under various conditions, to be the following multiples of the net section:

	Short spans	Long spans
Shallow girders,	$1\frac{3}{4}$ light loads to 2 for heavy loads	$2\frac{1}{4}$ to $2\frac{1}{2}$
Deep ,,	2 ,, $2\frac{1}{4}$,,	$2\frac{1}{2}$ to $3\frac{3}{4}$

As far as the web is concerned, there would obviously be a practical advantage in making the depth of a girder small in proportion to the span and load. Thus, in shallow girders heavily loaded the gross average thickness varies from twice the net for short spans, to two and a half times for long spans; but as the small depth is a disadvantage to the flanges, the determination of the depth at which the joint weights of the flanges and web will be a minimum is the problem to be solved.

Now, W being the distributed load, *t* the strain per square inch, the other notation remaining as before, the sectional area (a) at the centre of the flange, in square inches, will be $a = \frac{WS}{8at}$, which amount will also represent the actual horizontal section required in the half web. But we have seen that in practice this latter area is never less than ¾ S; consequently, the value of *a* for the web must never be taken at *less* than that amount. Again, since the horizontal section of the half web should theoretically be of triangular form, whereas in a web of uniform thickness it is of course rectangular, the value of the coefficient *y*, under the several conditions, will be double the multiples which we have deduced from the webs of existing girders. The respective amounts will therefore be as follows:

Shallow	girders,	3½	light loads	to 4	for heavy loads	Short spans	4½ to 5	Long spans
Deep	,,	4	,,	4½	,,		5 to 5½	

As we know the mass, and consequently the weight, of a girder to be proportioned to $a\,(Sx+yd)$, it is only necessary now to ascertain to what extent the limiting value of *a* for the web will affect the question.

Now, taking a web of uniform thickness, and adopting the highest value of *y* and the lowest of *x*, it is obvious that, if the mass of the flanges exceed that of the web, the depth must necessarily be too small, since an increased depth would similarly affect the weight of the web directly, and the flange inversely. Thus, assuming the mass of the flange to be to that of the web as 6 : 4, the sum being 10, then, if the depth were increased $\frac{1}{4}$, the mass of the web would be $\frac{4\times 5}{4}=5$ and that of the flange $\frac{6\times 4}{5}=4.8$, the sum being 9.8. Again, taking the highest value of x and the lowest of y, we can arrive in the same manner at the maximum depth. If the web be not of uniform thickness, it is even more apparent that an excess in the mass of the flanges over that of the web indicates deficient depth, since increased depth would involve a proportionally less increase in the weight of the web. We have, therefore, $ax\mathrm{S}=a_1yd$, and we know that a_1 can never be less than $\frac{3}{4}$ S, and never be more than $\frac{\mathrm{WS}}{8dt}$. Now, supposing that the thickness of the web might be reduced indefinitely, then, in order for the mass of the flange to be equal to that of the web, the span must obviously be to the depth as $y:x$; thus, theoretically, the flange and web will be of similar weights when the depth equals $\frac{2}{3}$ of the span. Although we cannot reduce the thickness of the web below a certain amount, the variations in the value of y are too small to affect materially the economy of using as thin a web as possible. Taking a web $\frac{1}{4}$ in. thick, therefore, we have $ax\mathrm{S}=\frac{3}{4}\mathrm{S}yd$, and $a=\frac{\mathrm{WS}}{8dt}$; therefore, $\mathrm{WS}x=6d^2ty$, and $d^2=\frac{\mathrm{WS}x}{6ty}$; or, substituting the *load per foot* (w), and the ratio of span to depth (r), we have for the proper ratio of span to depth,

$$r=\sqrt{\frac{6\,ty}{wx}}.$$

We have previously observed that a thicker plate than ¼ in. is desirable if the load per foot exceed a certain amount, the limits of which must of necessity be determined somewhat arbitrarily. This will, however, only affect the value of the constant (6), and we believe it will be sufficiently near the truth to assume this amount to remain constant up to a load of 30 cwt. per foot, and then to increase ½ for every additional 10 cwt. per foot, so that, at 100 cwt. per foot, it would be equal to 9.5.

Proceeding now to the consideration of girders with lattice webs, the first thing to be remarked is that whilst theoretically the weights of such webs must be at least double those required for plate webs of similar strength, yet practically, owing to the facility of approximating the section of lattice bars to the required amount, it is quite possible for this conclusion to be reversed. Since there is no practical difficulty in making the bars of a lattice web to any required section, it follows that in this instance there will not be a limiting value to a_1; consequently the most suitable depth of girder will not, as in the former instance, be governed chiefly by the minimum thickness of metal, but by the necessity of avoiding the employment of much metal in the disadvantageous form of struts or long columns. In deep light girders, where the triangulations are few, and consequently the effective length of strut great, this condition is very potent in determining the most suitable pattern of bracing. Thus, the proportional quantity of metal required for struts at various inclinations—the other conditions being the same for all cases—would, if the ratio of length to diameter were sufficient to cause failure by flexure, be as the cube of their respective lengths; consequently the weight of a vertical strut being 1, the weight of an equally strong strut at an angle of 45° would be $\sqrt[2]{8}=2.82$. Again, as the resistance to compression in such an instance would not be more than one half the resistance to tension, the weight of a complete triangulation, and consequently the proportional weight of the entire web, would, if we take the resistance to vertical tension as the unit, and the struts and ties at an angle of 45°, be $1^2+1^2+2\ (2.82)=7.64$; whilst if the struts were vertical, and the ties inclined at the rate of 2 to 1, the quantity required would be $2^2+1^2+2=7$, which is less than the former amount, and therefore reverses the theoretical deduction—namely, that the weight would be half as large again.

As there are so many conflicting conditions affecting the weight of a lattice web, we must necessarily, in this short paper, eliminate some by certain assumptions; we shall, therefore, in the expression previously given for the comparative weights of lattice girders:

$$\left(\mathrm{S}x+y\frac{l^2}{h}\right)$$

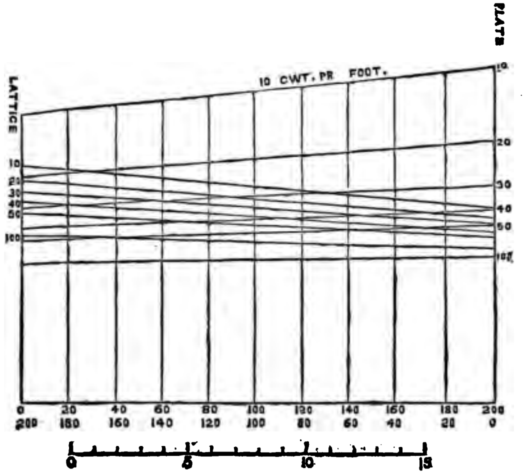
substitute $\frac{l^2}{h}=2$; that is, assume the bars to be at an angle of 45°; and we shall also assume that, within certain limits, the weight of the web will vary directly, and the weight of the flange inversely, as the depth. We have, therefore, $\mathrm{S}x=2yd$, and substituting $\frac{\mathrm{S}}{d}=r$, we have $r=\frac{2y}{x}$. Now the values of x will be those previously given for plate girders, but the values of y will, since there is no limiting value to a_1, have a much larger range. Thus, for shallow girders, heavily loaded, we find from actual examples, the value of y to be $=2$, whilst for deep girders lightly loaded, $y=8$.

Our investigation, therefore, results in the following formulæ for the most economical depth for girders:

$$\text{For plate girders, } r=\sqrt{\frac{6ty}{wx}}.$$

$$\text{For lattice girders, } r=\frac{2y}{x}.$$

We have also deduced the values of the coefficients x and y, within certain limits, from practical examples, but have not given a general expression applicable to any given case; because, even if it were possible to do so, the complexity of the formulæ would rather conceal than illustrate the relative importance of the different conditions affecting the case. We have, therefore, preferred computing the results for certain specific cases, and arranging them graphically in the form of a diagram. The figures on the base line of the diagram refer to the spans in feet, numbered from left to right for plate-girders, and from right to left for lattice; the figures on the end vertical lines refer to the load in cwts. per foot, the left-hand side for lattice and the right-hand side for plate girders. The height from the span on base line to the line corresponding to the load per foot, measured with the "vertical scale for ratio of span to depth," gives the most economical ratio of span to depth. The form the diagram assumes proves very satisfactorily the truth of our previous assumption, that the most important element governing the proper depth of a girder is the load per foot. The influence of the span is shown by the divergence of the lines from the horizontal, whilst the far more important influence of the load per foot is graphically illustrated by the divergence of the several lines from each other. As might have been predicted, the suitable depths for lattice girders under different loads are much more uniform than for plate girders; thus, whilst in the former case the ratios vary from 10 to 6¼, in the latter the range is from 14 to 5½.



In the present paper we have only attempted an approximate solution of a problem of considerable importance; the results arrived at are, however, sufficiently definite to suggest the advisability of giving greater depth to the average run of girders than is at present the practice in this country. Mr. Zerah Colburn, in a descriptive paper on "American Iron Bridges," read at the Institution of Civil Engineers, pointed out the different practice adopted in the United States in this respect. He says: "American engineers consider a depth of one-eighth of spans of 200 ft. as only moderately deep; for shorter spans depths of one-seventh and one-sixth are common; and in the case of one bridge of 120 ft. span, the depth from the centre of the top to the centre of the bottom chords is 23 ft.—a proportion of nearly one-fifth." Now, as American bridges may safely challenge comparison with ours, as far as the weight of metal and the factor of safety is concerned, notwithstanding the extravagant form of trussing usually adopted, we are quite justified in citing the above facts as corroborating the results arrived at in the preceding investigation.

PART VII.

FORMULÆ AND TABLES

Section 28.

FORMULÆ DERIVED FROM THE PRECEDING INVESTIGATIONS.

GENERAL equation for the breaking weight, applied at the centre, of a beam of any material; or of any form of cross section.

$$\text{B. W.}=\frac{f\text{M}\,(1+\varphi^1)}{4\text{S}}.$$

$$\text{When } \varphi^1=\frac{\varphi\times\text{thickness of “web” portion of cross section}}{\text{width of “flange” portion}}$$

The value of ϕ being (see Sec. 3—5) 1.25 for cast iron; .56 for wrought iron; from .6 to .8 for mild and strong steel respectively; and practically nothing for timber. And when S=the span in inches; M=the moment of resistance of the cross section, that is the effective depth of the beam in inches × the effective area of the "flange" portion in square inches (see Sec. 2), and f=the ultimate resistance of the material per square inch. When the value of fM for the parts in tension is less than for those in compression, the beam will fail by tension, and *vice versâ*. In the former instance, the strength of the beam may be computed to any required degree of accuracy; in the latter there will always be some difficulty in doing so, in consequence of the extent to which the ultimate resistance per square inch of the material in compression is governed by the *form* of the cross section, as well as by the relative areas of the portions in tension and compression. (See Secs. 4 and 8.)

For average cases the values of f may be taken as follows:

	Tension per square inch.	Compression per square inch.		
		(*a*)	(*b*)	(*c*)
	Tons.	Tons.	Tons.	Tons.
Cast iron	$f=7$	28	42	42
Wrought iron	$f=21$	16	21	28
Steel	$f=35$ to 50	25 to 30	35 to 50	45 to 65
Oak	$f=2\frac{1}{2}$ to $4\frac{1}{2}$	same as for tension.		
Pine	$f=2$ to $3\frac{1}{2}$		do.	do.

The lower value of f for steel will be that obtained in ordinary flanged rails, boiler plates, &c., and the first value of f should also be taken for timber of the class ordinarily used in permanent way. The column (*a*) refers to the compression members of beams with proportionally thin flanges; column (*b*) to those with thick flanges, and also to T-iron forms of cross section; and column (*c*) gives the approximate ultimate resistance to compression when the metal is in masses, as in the head of an ordinary rail.

Substituting the preceding values of f and ϕ in the general equation, we obtain the following:—

Formulæ for the breaking weight in tons at the centre of the several forms of bars enumerated below.

Section of Bar.	Cast Iron.	Wrought Iron.	Steel.	Oak.	Pine.
Rectangular.........	B.W.$=\frac{7}{8}\,\frac{d^2b}{L}$	$1\frac{3}{4}\,\frac{d^2b}{L}$	3 to 5 $\frac{d^2b}{L}$	$\frac{1}{7}$ to $\frac{1}{4}\,\frac{d^2b}{L}$	$\frac{1}{9}$ to $\frac{1}{5}\,\frac{d^2b}{L}$
Square in direction of diagonal	" $\frac{3}{4}\,\frac{d^3}{L}$	$1\frac{1}{2}\,\frac{d^3}{L}$	$2\frac{5}{8}$ to $4\frac{1}{4}\,\frac{d^3}{L}$	$\frac{1}{10}$ to $\frac{1}{6}\,\frac{d^3}{L}$	$\frac{1}{13}$ to $\frac{1}{7}\,\frac{d^3}{L}$
Round	" $\frac{9}{16}$ "	$1\frac{1}{8}$ "	2 to $3\frac{1}{4}$ "	$\frac{1}{12}$ to $\frac{1}{7}$ "	$\frac{1}{16}$ to $\frac{1}{9}$

B.W=breaking weight in tons at centre of bar; L=length of span in *feet;* and d=the depth of a rectangular bar, the side of a square one, or the diameter of a round one in *inches.*

Under a working load equal to ¼th the breaking weight at the centre in each instance the deflection in inches of the bars would be given by the following formulæ:

Section of Bar.	Cast Iron.	Wrought Iron.	Steel.	Oak or Pine.
Rectangular	$\delta = .017\,\frac{L^2}{d}$	$.019\,\frac{L^2}{d}$	$.026\,\frac{L^2}{d}$	$.034\,\frac{L^2}{d}$
Square in direction of diagonal..................	„ .015 „	.016 „	.022 „	.025
Round	„ .019 „	.021 „	.029 „	.034

Under a distributed load the deflections would be ¼th greater than the preceding.

The strain per square inch on the metal in a beam is generally a part of the *data*, but if not so, since the computation of the strength of a beam almost always precedes that of its deflection, the latter may be more conveniently expressed in terms of the strain per square inch than in terms of the load carried.

The deflection of any section of beam of the depth *d* in inches, and the span L in feet, under any given strain per square inch within the elastic limit, may be obtained from the following Table:

Distribution of Load.	Form of Beam.	Cast Iron.	Wrought Iron.	Steel.
Weight at centre......	No web, Uniform strain.	$\delta = .018\,\frac{L^2F}{d}$	$.0072\frac{L^2F}{d}$	$.0058\frac{L^2F}{d}$
„ „	„ thickness.	„ .0086	.0048	.0039
„ „	„ depth.	„ .0065	.0036	.0029
„ „	„ section.	„ .0043	.0024	.0019
Weight distributed ...	No web. Uniform strain.	„ .0086	.0048	.0039
„ „	„ thickness.	„ .0074	.0041	.0033
„ „	„ depth.	„ .0065	.0036	.0029
„ „	„ section.	„ .0054	.0030	.0024

F=the mean of the *maximum* apparent compression and tension in tons per square inch, that is in all cases,

$$F=\frac{2\,m}{M+M'},$$

when m=moment of the load, and M and M' the moments of resistance of the cross section in tension and compression respectively. (See Sec. 9.)

The deflection of uniformly loaded girders of the depth D in feet, under tensional strains of 1½ tons, 5 tons, and 8 tons per square inch, for cast iron, wrought iron, and steel respectively, will be given by the following formulæ:

Form of Girder.	Cast Iron Girder top flange ¼th area of bottom.	Wrought Iron.	Steel.
Uniform depth and section...	$\delta=.0016\,\frac{L^2}{D}$	$.00125\,\frac{L^2}{D}$	$.0016\,\frac{L^2}{D}$
Reduced	$\delta=.0019$ „	.0015 „	.0019

The deflection in inches of a girder under the preceding conditions, for each foot of the span is exhibited in the following Table:

Form of Girder.		Cast Iron.	Wrought Iron.	Steel.
Depth ⅛ span	Uniform depth and section...	.0125	.01	.0125
	Reduced „ „ ..	.015	.012	.015
Depth $\frac{1}{10}$ span	Uniform depth and section...	.016	.0125	.016
	Reduced „ „ ...	.019	.015	.019
Depth $\frac{1}{12}$ span	Uniform depth and section...	.019	.015	.019
	Reduced „ „ ...	.023	.018	.023

The camber necessary to be given to the angle irons of a girder in order that the underside of the flange at the centre may not deflect below the level of the upper surface of the bed plates may be obtained from the preceding Table by adding to the deflection there indicated the difference in thickness between the centre and end plates of the girder.

Girders designed for a distributed load are very commonly tested by hydraulic pressure applied at the centre of the span. Under proof strains of $2\frac{1}{2}$ tons, 5 tons, and 8 tons per square inch respectively, for cast iron, wrought iron, and steel girders so tested, the proof deflections would be as follows:

Form of Girder.	Cast Iron Girder top flange $\frac{1}{4}$th area of bottom.	Wrought Iron.	Steel.
Uniform depth and section...	$\delta = .0022 \frac{L^2}{D}$	$.001 \frac{L^2}{D}$	$.00133 \frac{L^2}{D}$
Reduced	$\delta = .0028$ „	.00125 „	.00166

The ultimate deflections of beams of uniform section, tested by pressure applied at the centre of the bearings, are expressed by the following equations:

$$\text{Cast iron beams } \delta = .000035 \frac{S^2 F}{d}.$$

$$\text{Wrought iron and steel } \delta = \frac{S^2 (e \times FE)}{12d}.$$

When S=the span and d, the depth, both in inches; E=.0001 for wrought iron, and .00008 for steel; and e=from .05 to .25 for wrought iron; and from .05 to .20 for steel. (See Secs. 8 and 9.)

The additional angle through which a solid bar of moderate size may be bent will be given by the equation,

$$\theta = \frac{180^\circ}{\pi d}$$

for ordinary fibrous iron, to [fourfold that amount for the toughest make of bars. (See Sec. 8.)

The deflection of a compound beam of iron or steel and wood may be obtained by substituting the following value of F in either of the general formulæ for the deflection of homogeneous beams:

$$F=\frac{m}{M+\frac{M^1d^1}{c\,d}}.$$

When m=moment of the weight in inch tons; M the moment of resistance of the metal portion of the beam, and M′ that of the timber portion; d and d' the depth in inches of the same members respectively; and the constant c=30 for an iron and pine beam.
22 for an iron and oak beam.
37 for a steel and pine beam.
27 for a steel and oak beam. (See Sec. 12.)

ARCHED RIBS.

A = sectional area in square inches at the springing.
A^1= " " " centre.
A^2= " " " haunch.

For round-ended arched ribs.

$$A=\frac{a}{t}.$$

$$A^1=\frac{a(b+1)}{2t-\frac{2rx}{d}}.$$

$$A^2=\frac{ab}{t-\frac{3rx}{4d}}.$$

For square-ended arched ribs.

$$A=\frac{ac}{t-\frac{15rx}{8d}}.$$

$$A^{1\cdot\cdot2}=\frac{a(c+1)}{2t-\frac{3rx}{d}}.$$

t=tons per square inch on the metal; W=total distributed load in tons; w=ratio of total load to rolling load; r=ratio of span to rise of arch;

$$a=\frac{Wr}{8}\sqrt{1+\frac{16}{r^2}};$$

$$b=\frac{\frac{d}{4r}+\frac{3}{4}+w-1}{w};$$

$$c=\frac{\frac{d}{5r}+\frac{3}{4}+w-1}{w};$$

and the value of x, in England,=1.62 for cast iron, 2.75 for cast iron, and 3.44 for steel; and in countries, such as America, where there are great extremes of temperature, about 2½ times those amounts.

Section 29.

CONTINUOUS GIRDERS.

The determination of the exact mathematical value of the maximum strains occurring in continuous girders of three or more spans, when subjected to a moving load, involves problems of almost hopeless intricacy. The most expert mathematicians would have to devote a month or more to the preliminary calculations for a very ordinary bridge, and the results deduced would not, after all, be more reliable in practice than those arrived at by comparatively simple modes of investigation, chiefly on account of the varying elasticity of different portions of even the same plate of iron.

The following Tables, based upon the laborious investigations of Bellanger, Laissle, Schübler, and others, are perfectly reliable for all practical purposes. The maximum sectional areas required at different points of the continuous girders are expressed in terms of that required at the centre of an independent girder of a span equal to the largest one occurring in the continuous girder. It may be observed here, that when the section of the

continuous girder, as designed, is found to vary considerably at different points, it will be well to throw an additional amount of sectional area, not exceeding 5 per cent., into the portions over the intermediate piers. On the other hand, when the intermediate piers are of considerable width, and the girder, at the same time, is of tolerably uniform section, a similar increase of section should be provided for in the central portions of the girder between the supports.

The diagrams representing the maximum sectional area required at any point in a continuous girder under the given distribution of load, may be constructed by setting off (see Fig. 39) the sectional area (*a*) required at the centre of an independent girder of the span S, to such a scale that the height so set off may not much exceed ⅛th of the span S, within which limits a circular arc and a parabola are practically identical. To the same scale, then, set up the heights *b*, *c*, *d*, &c., derived from the following Tables, and connect those points by curved lines, as shown in full on the diagram, the common radius of which will be the same as that of the curve of which the chord line is the span S, and the ver. sin. the height *a*. The vertical height at any point of the span will then represent the maximum sectional area required at that point in a continuous girder of uniform depth.

Continuous girders of two equal spans.

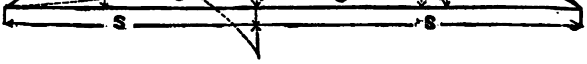


FIG. 39.

$a=1.000$. W = the minimum distributed load per foot run, divided by the maximum distributed load per foot run, on the girders; or, in other words, the proportion of dead load to the total load.

W	*b*	*c*	*d*	*c'*	*d'*
1	1.000	.560	.000	.560	.000
$\frac{7}{8}$	"	.584	.065	.584	.065
$\frac{3}{4}$	"	.608	.110	.608	.110
$\frac{5}{8}$	"	.632	.160	.632	.160
$\frac{1}{2}$	"	.656	.205	.656	.205
$\frac{3}{8}$	"	.682	.247	.682	.247
$\frac{1}{4}$	"	.710	.287	.710	.287
$\frac{1}{8}$	"	.738	.323	.738	.323

Continuous girders of three or more spans with the end spans, as desirable, eight-tenths of the intermediate ones.

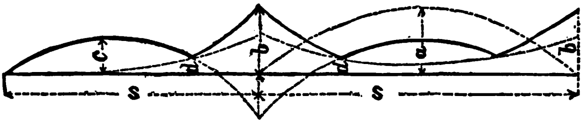


FIG. 40.

W	*b*	*c*	*d*	*c'*	*d'*
1	.660	.352	.000	.344	.000
$\frac{7}{8}$	.690	.373	.052	.371	.040
$\frac{3}{4}$	.700	.394	.086	.398	.062
$\frac{5}{8}$	.710	.416	.121	.425	.085
$\frac{1}{2}$	.720	.438	.157	.453	.109
$\frac{3}{8}$	.730	.461	.195	.481	.135
$\frac{1}{4}$	.740	.484	.235	.509	.163
$\frac{1}{8}$	.750	.508	.284	.538	.195

Continuous girders of three equal spans.

W	b	c	d	c'	d'
1	.800	.640	.000	.200	.000
$\frac{7}{8}$	.816	.660	.040	.260	.090
$\frac{3}{4}$	.832	.680	.179	.315	.134
$\frac{5}{8}$	.849	.700	.119	.365	.179
$\frac{1}{2}$	.866	.721	.160	.410	.225
$\frac{3}{8}$	.883	.742	.202	.455	.272
$\frac{1}{4}$	.900	.763	.245	.495	.320
$\frac{1}{8}$	.917	.785	.290	.530	.370

Continuous girders of two unequal spans.

$\frac{s}{S}$ = ratio of short span to long span.

VALUE OF c.

W	$\frac{s}{S}=1$	$=\frac{7}{8}$	$=\frac{3}{4}$	$=\frac{5}{8}$	$=\frac{1}{2}$
1	.560	.360	.212	.90	.015
$\frac{7}{8}$	.584	.382	.231	.99	.032
$\frac{3}{4}$	.608	.404	.250	.118	.050
$\frac{5}{8}$	.532	.426	.270	.137	.068
$\frac{1}{2}$	.656	.448	.290	.156	.086
$\frac{3}{8}$	.682	.470	.310	.175	.104
$\frac{1}{4}$	.710	.493	.330	.200	.122
$\frac{1}{8}$	.738	.516	.350	.225	.140

VALUE OF b.

W	$\frac{s}{S}=1$	$=\frac{7}{8}$	$=\frac{3}{4}$	$=\frac{5}{8}$	$=\frac{1}{2}$
1 to 0	1.000	.940	.880	.820	.760

VALUE of c'.

W	$\frac{s}{S}=1$	$=\frac{7}{8}$	$=\frac{3}{4}$	$=\frac{5}{8}$	$=\frac{1}{2}$
1	.560	.585	.610	.635	.660
$\frac{7}{8}$	.584	.607	.629	.651	.672
$\frac{3}{4}$	.608	.629	.648	.667	.685
$\frac{5}{8}$	.632	.651	.668	.683	.698
$\frac{1}{2}$	.656	.674	.688	.699	.711
$\frac{3}{8}$	.682	.697	.708	.715	.724
$\frac{1}{4}$	.710	.720	.728	.732	.737
$\frac{1}{8}$	.738	.743	.748	.749	.750

Continuous girders of three spans.

$\frac{s}{S}$ = ratio of side spans to centre span.

VALUE OF c.

W	$\frac{s}{S}=1$	$=\frac{7}{8}$	$=\frac{3}{4}$	$=\frac{5}{8}$	$=\frac{1}{2}$
1	.640	.450	.280	.140	.040
$\frac{7}{8}$	.660	.469	.297	.156	.055
$\frac{3}{4}$	.680	.488	.314	.172	.070
$\frac{5}{8}$	.700	.507	.332	.188	.086
$\frac{1}{2}$	.721	.526	.350	.205	.102
$\frac{3}{8}$	.742	.545	.368	.222	.118
$\frac{1}{4}$	.763	.565	.386	.239	.134
$\frac{1}{8}$	.785	.585	.404	.256	.150

VALUE OF b.

W	$\frac{s}{S}=1$	$=\frac{7}{8}$	$=\frac{3}{4}$	$=\frac{5}{8}$	$=\frac{1}{2}$
1	.800	.703	.631	.585	.562
$\frac{7}{8}$	.816	.716	.640	.591	.566
$\frac{3}{4}$	.832	.729	.650	.597	.570
$\frac{5}{8}$	.849	.742	.660	.603	.574
$\frac{1}{2}$	.866	.754	.669	.609	.578
$\frac{3}{8}$	.883	.766	.678	.615	.582
$\frac{1}{4}$	.900	.779	.688	.621	.586
$\frac{1}{8}$	.917	.793	.698	.627	.590

VALUE OF c'.

W	$\frac{s}{S}=1$	$=\frac{7}{8}$	$=\frac{3}{4}$	$=\frac{5}{8}$	$=\frac{1}{2}$
1	.200	.297	.369	.415	.438
$\frac{7}{8}$	.260	.338	.392	.427	.448
$\frac{3}{4}$	.315	.376	.415	.440	.459
$\frac{5}{8}$	.365	.411	.438	.455	.471
$\frac{1}{2}$	.410	.444	.462	.472	.485
$\frac{3}{8}$	.455	.475	.486	.491	.501
$\frac{1}{4}$	.495	.505	.512	.515	.519
$\frac{1}{8}$	.530	.534	.538	.539	.540

It is not considered necessary to tabulate the values of d and d', as the preceding Tables afford sufficient *data* for obtaining the same by construction. Thus, referring to the preceding diagrams, the process will be as follow:

Continue the curve of the central portion of the girder till it intersects the pier below the base line, as shown in dotted lines on the diagrams. Set off a corresponding distance above the base line, and draw in the dotted curved lines shown on the diagrams, the radii of which will be equal to the radius of the curve for the central portion of the diagram divided by the fraction W. Then the intersection of these dotted curves with the full line curves will give the points d and d'.

Section 30.

Weights of wrought-iron plate girders. Depth $\frac{1}{10}$th span, working strain 4½ tons per square inch in tension.

LOAD PER FOOT RUN (exclusive of the weight of the girders).

SPAN IN FEET.	10	15	20	25	30	35	40	50	60	70	80
	WEIGHTS OF GIRDERS IN CWT.										
10	5.6	6.5	7.4	8.3	9.2	10.2	11	12.8	14.6	16.2	17.5
15	10	11.9	13.5	15.2	16.7	18.4	20.2	23.4	26.4	29.2	32
20	17.8	20.5	23.4	26.3	29	31.7	34.5	40.3	45.6	50.5	55.5
25	26.6	31	35.4	39.3	44.2	48	52	60.8	68.6	76	83.5
30	38	44	50	56	62	68	74	86	97	108	118
35	51	58	66	74	82	89	97	113	129	145	160
40	65	75	85	95	105	115	125	145	166	187	209
45	82	94	106	118	130	142	154	180	207	236	264
50	101	115	130	144	159	173	188	220	254	290	325
55	118	135	152	169	187	204	222	259	298	340	382
60	138	157	177	196	217	237	258	302	348	395	441
65	159	181	204	227	251	275	298	348	400	453	507
70	198	225	253	282	312	342	372	435	500	565	630
75	228	260	292	326	360	394	429	500	575	650	726
80	258	294	331	369	407	446	484	566	650	735	822
85	291	333	375	416	460	502	546	637	732	829	928
90	326	373	420	467	515	563	612	712	818	927	1040
95	365	417	470	523	576	630	686	800	920	1043	1172
100	406	464	522	581	641	701	764	892	1028	1167	1310
110	495	565	636	708	780	855	930	1090	1260	1430	1610
120	595	677	762	848	934	1020	1112	1305	1510	1720	1940
130	705	800	900	1000	1100	1200	1310	1540	1780	2000	2300
140	823	940	1059	1178	1298	1417	1546	1810	2085	2376	2686
150	950	1090	1230	1370	1510	1650	1800	2100	2410	2740	3100
160	1095	1255	1414	1574	1732	1896	2066	2415	2782	3172	3585
170	1250	1430	1610	1790	1970	2160	2350	2750	3180	3630	4100
180	1426	1626	1826	2036	2240	2450	2670	3140	3630	4130	4670
190	1614	1832	2060	2280	2510	2750	3010	3550	4100	4670	5270
200	1810	2050	2300	2550	2800	3070	3370	3980	4600	5230	5900

The preceding Table was constructed by taking as far as possible the weights of girders actually erected, calculating missing links in the series, rectifying the curves, and interpolating. Any general formula for the weights of girders must necessarily be very complicated; because not only does the weight of the girder itself constitute a portion of the load, but the percentage of useless metal on account of the requirements of manufacture varies considerably with the intensity of the load per foot; consequently, the weights of differently loaded girders of a given span are by no means constant functions of the load. Thus, referring to the preceding Table, it will be seen that in the instance of the 200 ft. girders, *three* times the quantity of metal required to carry ½ a ton per foot run will suffice to carry *eight* times the load. The same kind of variation takes effect in different degrees until the load attains about 70 cwt. per foot run. Above that limit the weight of girder may be taken as a constant function of the load, thus the weight of iron required to carry 120 cwt. per foot run would be one and a half times that given in the Table for 80 cwt. per foot. The weights of girders in terms of the load, between the limits of 10 ft. to 200 ft. span, and 10 cwt. to 70 cwt. per foot run, may be represented by a certain solid. Let two parallel sides of the base of a cube represent the spans between 10 ft. and 200 ft., and the two remaining sides the loads per foot from 10 cwt. to 70 cwt. Also let the vertical height of the cube at the corner corresponding to 10 cwt. per foot, and 200 ft. span, represent the weight of the girder in terms of the load; and let the height of the other corners of the solid represent corresponding percentages. Then if the top of the cube be cut away between these four corners to a double curve surface of a certain order we shall have a solid, the thickness of which at any point will represent the corresponding weight of the girder in terms of the load. The position of any point in the curve surface will be given by the three rectangular co-ordinates x, y, z, functions respectively of the span, and load per foot, and weight of the girder in terms of the useful load. The equation of this curve surface derived from the *data* afforded by the "Table of Weights" is very nearly as follows:

$$(x^2+.02x+.04)\,1.348-\sqrt[4]{.0145\,(z-.1357)\,(1.5714-z)}\;-y=0$$

From this equation formulæ may readily be derived for the weights of girders; but they would, necessarily, be somewhat clumsy in application.

THE END.

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